Complex Numbers

Preben Alsholm

September 4, 2008

1 Complex Numbers

1.1 Sets of numbers

Sets of numbers

- N is the set of natural numbers, $1, 2, 3, 4, 5, \ldots$
- Z is the set of integers $\ldots - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$.
- Q is the set of rational numbers, i.e. fractions and integers.
- R is the set of real numbers. Can be identified with the set of points on a straight line, the real number line. *Irrational* numbers are real numbers that are not rational.
- C is the set of complex numbers. Can be identified with the set of points in the plane, the complex plane. Complex numbers that are not real are called *imaginary*.
- We have $N \subset Z \subset Q \subset R \subset C$.

1.2 Rules for addition and multiplication

Rules for addition and multiplication

1. $a + b = b + a$ (the commutative law of addition)
2. $(a + b) + c = a + (b + c)$ (the associative law of addition)
3. $ab = ba$ (the commutative law of multiplication)
4. $(ab)\, c = a\, (bc)$ (the associative law of multiplication)
5. $a\, (b + c) = ab + ac$ (the distributive law)
6. $a + 0 = a$
7. $1a = a$
8. $a + x = 0$ has precisely one solution for x
9. $ax = 1$ has precisely one solution for x, provided $a \neq 0$
10. Every Cauchy sequence has a limit

1.3 What is a Cauchy sequence?

What is a Cauchy sequence?

- A *Cauchy sequence* is a sequence of numbers $(x_n)_{n=1}^{\infty} = x_1, x_2, x_3, \ldots, x_n, \ldots$ satisfying $x_n - x_m \to 0$ for $n, m \to \infty$.
- That every Cauchy sequence has a limit means that $x_n - x_m \to 0$ for $n, m \to \infty$ implies that there exists a number x, such that $x_n \to x$ for $n \to \infty$.
- The claim: *Every Cauchy sequence has a limit* is valid for R and C, not for Q.
- C has the very important property that *every polynomial of degree* ≥ 1 *has at least one root*. (The Fundamental Theorem of Algebra).

2 Description of the complex numbers

Description of the complex numbers

- As a set C equals the set of points in the plane. The plane is identified with R^2, thus $C = R^2$.
- The point (a_1, a_2) is written $a_1 + a_2 i$. Thus i is the point $(0, 1)$ and 1 is the point $(1, 0)$.
- Definition of addition. If $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ then $a + b = (a_1 + b_1) + (a_2 + b_2)\, i$.
- Definition of multiplication. If $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ then

$$ab = (a_1 + a_2 i)(b_1 + b_2 i) = (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1)\, i$$

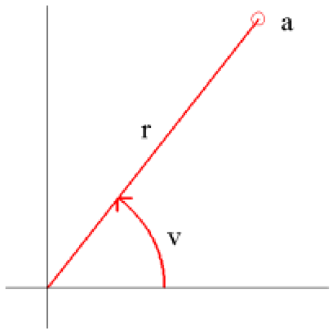
- It follows that $i^2 = -1$.

2.1 Division?

Division?

- The solution to the equation $az = 1$ exists if $a \neq 0$ and is uniquely determined. It is denoted a^{-1} or $\frac{1}{a}$.
- By $\frac{b}{a}$ we mean ba^{-1}. It is the solution to the equation $az = b$
- The usual method of calculating $\frac{b}{a}$:

$$\begin{aligned}\frac{2+3i}{-4+7i} &= \frac{(2+3i)(-4-7i)}{(-4+7i)(-4-7i)} = \frac{(2+3i)(-4-7i)}{(-4)^2 - (7i)^2} \\ &= \frac{(2+3i)(-4-7i)}{16+49} = \frac{13-26i}{65} = \frac{1}{5} - \frac{2}{5}i\end{aligned}$$



2.2 Real and imaginary parts etc.

Real and imaginary parts etc.

- Real part: $\text{Re}\,(a_1 + ia_2) = a_1$. Imaginary part: $\text{Im}\,(a_1 + ia_2) = a_2$
- Complex conjugate: $\overline{a} = \overline{a_1 + ia_2} = a_1 - ia_2$
- $\overline{a+b} = \overline{a} + \overline{b}$ and $\overline{(ab)} = \overline{a}\overline{b}$
- Modulus, absolute value: $|a| = |a_1 + ia_2| = \sqrt{a_1^2 + a_2^2}$
- $|ab| = |a|\,|b|$
- The triangle inequality: $|a+b| \leq |a| + |b|$

2.3 Polar form I

Polar form I

- Let $r = |a|$ and v be an angle measured from the positive real axis to the line connecting 0 and a (measured positive in the counterclockwise direction).
- v is an argument for a. Notation: $\arg\,(a)$. The set of arguments for a is $\{v + p2\pi\,|p \in Z\}$.
- Any complex number can be written in polar form: $a = r \cdot (\cos v + i \sin v)$, where r is the modulus and v is an argument of a.

2.4 Polar form II

Polar form II

- The principal value: $Arg\,(a)$ is the uniquely given argument in the interval $]-\pi, \pi]$.
- By $\arg_\tau\,(a)$ we mean the unique argument in the interval $]\tau, \tau + 2\pi]$, thus $Arg\,(a) = \arg_{-\pi}\,(a)$.
- $\arg\,(ab) = \arg a + \arg b$

- $\arg(a^n) = n \arg a$
- $\arg\left(\frac{a}{b}\right) = \arg a - \arg b$
- These must be properly understood: Thus $\arg(ab) = \arg a + \arg b$ means that one of the arguments for ab is obtained by adding an argument for a and an argument for b.

2.5 The Complex Exponential

The Complex Exponential

- The real exponential function exp has the fundamental property

$$\exp(x+y) = \exp(x)\exp(y)$$

  i.e. $e^{x+y} = e^x e^y$ for all $x, y \in R$.

- Definition. If $z = x + iy$ $(x, y \in R)$ then

$$\exp(z) = \exp x \cdot (\cos y + i \sin y)$$

- $\left|e^{x+iy}\right| = e^x$ and $\arg\left(e^{x+iy}\right) = y$ when $x, y \in R$.
- $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ for all $z_1, z_2 \in C$, i.e. $e^{z_1+z_2} = e^{z_1}e^{z_2}$.
- Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\left|e^{z_1} \cdot e^{z_2}\right| &= \left|e^{z_1}\right| \cdot \left|e^{z_2}\right| = \left|e^{x_1+iy_1}\right| \cdot \left|e^{x_2+iy_2}\right| = e^{x_1} \cdot e^{x_2} \\
&= e^{x_1+x_2} = \left|e^{x_1+x_2+i(y_1+y_2)}\right| = \left|e^{z_1+z_2}\right| \\
\arg\left(e^{z_1} \cdot e^{z_2}\right) &= \arg\left(e^{z_1}\right) + \arg\left(e^{z_2}\right) = \arg\left(e^{x_1+iy_1}\right) + \arg\left(e^{x_2+iy_2}\right) \\
&= y_1 + y_2 = \arg\left(e^{x_1+x_2+i(y_1+y_2)}\right) = \arg\left(e^{z_1+z_2}\right)
\end{aligned}$$

2.6 The polar form once more

The polar form once more

- The polar form for the number a having modulus r and argument v was written

$$a = r(\cos v + i \sin v)$$

  In the future we shall write:

$$a = r\exp(iv) = re^{iv}$$

- Example. The polar form for $-\sqrt{3} - i$. Modulus: $\sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$. An argument is $-\frac{5\pi}{6}$. Thus

$$-\sqrt{3} - i = 2\exp\left(-i\frac{5\pi}{6}\right) = 2e^{-i\frac{5\pi}{6}}$$

2.7 De Moivre's formula

De Moivre's formula

- For $n \in N$ og $\theta \in R$ gælder

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- Proof: $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$
- Example.

$$\begin{aligned}
\cos 3x &= \mathrm{Re}\left(\cos 3x + i\sin 3x\right) = \mathrm{Re}\left((\cos x + i\sin x)^3\right) \\
&= \mathrm{Re}\left(\cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x\right) \\
&= \cos^3 x - 3\cos x\sin^2 x = \cos^3 x - 3\cos x\left(1 - \cos^2 x\right) \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

- By replacing Re with Im above we get the formula

$$\begin{aligned}
\sin 3x &= 3\cos^2 x\sin x - \sin^3 x \\
&= 3\left(1 - \sin^2 x\right)\sin x - \sin^3 x \\
&= -4\sin^3 x + 3\sin x
\end{aligned}$$