Week 3, 2.3 - 2.4, and 3.1

Preben Alsholm

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1 Analyticity. Polynomials and Rational Functions.

1.1 Differentiability

Differentiability

- Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$. Let z_0 be interior to A. If the limit

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists, then f is *differentiable* at z_0. The value of the limit is the *derivative* of f at z_0 and is denoted by $f'(z_0)$.

- Notice that h in $f(z_0+h)$ above is complex.
- Example 2: The function $z \to \bar{z}$ is not differentiable anywhere.
- Example 3: The function $z \to z^n$ ($n \in \mathbb{N}$) is differentiable everywhere, and $\frac{d}{dz}z^n = nz^{n-1}$.
- Proof: The binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

- Thus $\left((z+h)^n - z^n\right)h^{-1} = \sum_{k=1}^{n} \binom{n}{k} z^{n-k}h^{k-1} \to \binom{n}{1}z^{n-1} = nz^{n-1}$ as $h \to 0$.

1.2 Differentiability. Analyticity.

Differentiability. Analyticity.

- Sums and products of functions differentiable at z are differentiable at z.
- If f and g are differentiable at z then $\frac{f}{g}$ is differentiable at z if $g(z) \neq 0$.
- The *chain rule*: If g is differentiable at z and f is differentiable at $g(z)$ then $f \circ g$ is differentiable at z and $(f \circ g)'(z) = f'(g(z))\,g'(z)$.
- Definition. Let $G \subseteq \mathbb{C}$ be open. If $f : G \to \mathbb{C}$ is differentiable at all points of G then f is *analytic* in G.

- Instead of "*analytic*" the word "*holomorphic*" is often used.
- " f is analytic at z_0 " means that f is analytic in a neighborhood of z_0.
- If f is not analytic at a point z_0 which is an accumulation point for points at which f is analytic, then z_0 is *singular point* for f.
- If f is analytic in all of $\mathbb{C}$ then f is an *entire* function.

1.3 About differentiability of $f : \mathbb{R}^2 \to \mathbb{R}^2$

About differentiability of vector functions

- Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$. Define the function f_R defined on a subset of $\mathbb{R}^2$ and having values in $\mathbb{R}^2$ by:

$$f_R(x, y) = \begin{bmatrix} \operatorname{Re} f(x + iy) \\ \operatorname{Im} f(x + iy) \end{bmatrix}$$

- A function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) if there exists a 2×2-matrix M s.t.

$$\frac{f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) - Mh}{\|h\|} \to 0$$

as $h \to 0$ (here 0 and h are vectors). The matrix M is the Jacobian matrix (sometimes denoted by $Df(x_0, y_0)$) and is given by

$$Df(x_0, y_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

where the partials are taken at (x_0, y_0).

1.4 The Cauchy-Riemann Equations I

The Cauchy-Riemann Equations I

- If $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 = x_0 + iy_0$ with derivative $f'(z_0) = a + ib$, then f_R is differentiable at (x_0, y_0) and

$$Df_R(x_0, y_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- The *Cauchy-Riemann equations*. If $f(x + iy) = u(x, y) + iv(x, y)$ then

$$Df_R = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

- Conversely, if $f_R = \begin{bmatrix} u \\ v \end{bmatrix}$ is differentiable at (x_0, y_0) and if the Cauchy-Riemann equations hold at that point, then $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0$ and $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

1.5 Proofs

Proofs

- We show that $f : A \subseteq \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 = x_0 + iy_0$ with derivative $f'(z_0) = c = a + ib$, iff f_R is differentiable at (x_0, y_0) and

$$Df_R(x_0, y_0) = A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- Differentiability of f at z_0 means that

$$\frac{f(z_0 + h) - f(z_0) - ch}{h} \to 0 \text{ as } h \to 0$$

- This statement is equivalent to

$$\frac{f(z_0 + h) - f(z_0) - ch}{|h|} \to 0 \text{ as } h \to 0$$

- With $\mathbf{0}$ being the zero vector in $\mathbb{R}^2$ this is equivalent to

$$\frac{f_R(x_0 + h_1, y_0 + h_2) - f_R(x_0, y_0) - Ah_R}{\|h_R\|} \to \mathbf{0} \text{ as } h_R = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \to \mathbf{0}$$

1.6 The Cauchy-Riemann Equations II

The Cauchy-Riemann Equations II

- Sufficient for a function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ to be differentiable at (x_0, y_0) is that its first partial derivatives exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0).
- Thus a sufficient condition for a function $f : A \subseteq \mathbb{C} \to \mathbb{C}$ ($f(x + iy) = u(x, y) + iv(x, y)$) to be differentiable at a point $x_0 + iy_0$ is that the first partial derivatives of u and v exist in a neighborhood of (x_0, y_0), are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0).
- exp is an entire function and satisfies $\frac{d}{dz}e^z = e^z$ for all $z \in \mathbb{C}$.
- If f is analytic in a domain D and if $f' = 0$ in D, then f is constant in D.

1.7 The Cauchy-Riemann Equations (an example)

The Cauchy-Riemann Equations (an example)

- Let f be analytic in the domain D. If $|f|$ is constant in D, then f is constant in D.
- Proof: $|f| = c \Rightarrow |f|^2 = u^2 + v^2 = c^2$. If $c = 0$ then $u = v = 0$ so $f = 0$.

- If $c > 0$ we differentiate $u^2 + v^2 = c^2$ w.r.t. x and y and get

$$\begin{aligned} u \cdot u_x + v \cdot v_x &= 0 \\ u \cdot u_y + v \cdot v_y &= 0 \end{aligned}$$

- Since $(u, v) \neq (0, 0)$ we conclude

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = 0$$

- Thus $0 = u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2$ so that $v_x = v_y = u_x = u_y = 0$.

- Thus u and v are constant, so f is.

1.8 Polynomials and Rational Functions I

Polynomials and Rational Functions I

- *Polynomial* in z over $\mathbb{C}$: $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_j \in \mathbb{C}$.

- *Rational function* $R(z)$: A ratio of two polynomials: $R(z) = \frac{p(z)}{q(z)}$.

- Polynomials are entire functions. Rational functions are analytic everywhere except at the zeros of the denominator.

- If $p(z)$ is a polynomial of degree n and $q(z)$ a polynomial of degree m with $1 \leq m \leq n$ then by the division algorithm we can write

$$p(z) = q(z) d(z) + r(z)$$

  where $\deg(r(z)) < \deg(q(z))$.

- See Maple illustration: PolyDiv. Factorization. The *Fundamental Theorem of Algebra*: Every nonconstant polynomial with complex coefficients has at least one zero (i.e. root) in $\mathbb{C}$.

1.9 Polynomials and Rational Functions II

Polynomials and Rational Functions II

- Consider a rational function $R(z)$. Cancel common factors of numerator and denominator.

- The zeros of the denominator are the *poles* of the rational function.

- Partial fraction expansion of a rational function. If the denominator contains the factor $(z - \zeta)^d$ then the expansion contains the terms

$$\frac{A_{d-1}}{z - \zeta} + \frac{A_{d-2}}{(z - \zeta)^2} + \ldots + \frac{A_0}{(z - \zeta)^d}$$

- The coefficients A_k are the coefficients in the $d-1$'st Taylor polynomial $T(z)$ of $g(z) = (z-\zeta)^d R(z)$ about ζ. Thus

$$A_k = \lim_{z \to \zeta} \frac{1}{k!} \frac{d^k}{dz^k} \left((z-\zeta)^d R(z) \right)$$

- In the proof we shall use that $g^{(k)}(\zeta) = T^{(k)}(\zeta)$ for $k = 0, 1, \ldots, d-1$.

1.10 Proof of the expansion

Proof of the expansion

- We may write $R(z) = \frac{P(z)}{(z-\zeta)^d Q(z)}$. Thus $g = \frac{P}{Q}$ and $(g-T)Q = P - QT$.
- $(g-T)Q = 0$ at ζ. If $d > 1$ its derivative $(g-T)' Q + (g-T) Q'$ is also zero.
- In fact all derivatives up to an including the $d-1$st are zero at ζ.
- Thus the polynomial $P(z) - Q(z)T(z)$ factors into $(z-\zeta)^d p(z)$, where p is a polynomial.
- We conclude that $R(z) - \frac{T(z)}{(z-\zeta)^d} = \frac{g-T}{(z-\zeta)^d} = \frac{p(z)}{Q(z)}$ has no pole at ζ.
- Repeat the procedure above on $\frac{p(z)}{Q(z)}$ using the next pole etc. We are left with

$$R(z) - \sum_{j=1}^{r} \frac{T_j(z)}{\left(z-\zeta_j\right)^{d_j}} = poly$$

- That *poly* must be the zero polynomial (consider $z \to \infty$).