Week 4, 3.2, 3.3, and 3.5

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1 The Exponential, Trigonometric, and Hyperbolic Functions

1.1 The Exponential and Trigonometric Functions

The Exponential and Trigonometric Functions

- *Definition.* $\exp : \mathbb{C} \to \mathbb{C}$ is defined by $e^z = e^{x+iy} = e^x (\cos y + i \sin y)$.
- exp is an entire function satisfying $\frac{d}{dz} e^z = e^z$ for all $z \in \mathbb{C}$.
- When $z = x + iy$, and $x, y \in \mathbb{R}$ we have $|e^z| = e^x$ and $\arg e^z = y + p2\pi, p \in \mathbb{Z}$.
- Theorem. $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = p2\pi i$ for some $p \in \mathbb{Z}$.
- exp is periodic with period $2\pi i$.
- Euler's formulas: $\cos y = \frac{1}{2}\left(e^{iy} + e^{-iy}\right), \sin y = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right)$.
- *Definition.* $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$.
- sin and cos are entire functions and $\frac{d}{dz} \cos z = -\sin z$, $\frac{d}{dz} \sin z = \cos z$.
- The well-known identities known from the real trig. functions still hold.

1.2 The Trigonometric and Hyperbolic Functions

The Trigonometric and Hyperbolic Functions

- The zeros of the complex cos and sin are its well-known real zeros.
- *Definition.* $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$, $\sec z = \frac{1}{\cos z}$, $\csc z = \frac{1}{\sin z}$.
- $\tan, \cot, \sec$, and $\csc$ are analytic in $\mathbb{C}$ except at the zeros of the respective denominators.
- *Definition.* The hyperbolic functions are defined in the usual way: $\sinh z = \frac{1}{2}\left(e^z - e^{-z}\right)$ and $\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right)$. But now we allow $z \in \mathbb{C}$.
- sinh and cosh are entire functions and $\frac{d}{dz} \cosh z = \sinh z$, $\frac{d}{dz} \sinh z = \cosh z$.
- *Definition.* $\tanh z = \frac{\sinh z}{\cosh z}$, $\coth z = \frac{\cosh z}{\sinh z}$,
  $\operatorname{sech} z = \frac{1}{\cosh z}$, $\operatorname{csch} z = \frac{1}{\sinh z}$.

1.3 The Logarithmic Function

The Logarithmic Function

- *Definition.* $w \in \mathbb{C}$ is a *logarithm* of $z \in \mathbb{C}$ if $e^w = z$.
- Every $z \in \mathbb{C} \setminus \{0\}$ has a logarithm, in fact infinitely many.
- The logarithms of $z \in \mathbb{C} \setminus \{0\}$ are given by

$$\log z = \ln|z| + i \arg z = \ln|z| + i Arg\ z + p2\pi i,\ \ p \in \mathbb{Z}$$

  where ln is the well-known real-valued logarithm defined on $\mathbb{R}_+$, and where Arg is the principal argument.
- log is an example of a *multiple-valued* function, as is arg.
- Just like $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ has to be understood correctly so does $\log(z_1 z_2) = \log z_1 + \log z_2$.
- The *principal value* of the logarithm is $Log\ z = \ln|z| + i Arg\ z$.
- Theorem. Log is analytic on $D^* = \mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$ and $\frac{d}{dz} Log\ z = \frac{1}{z}$ for all $z \in D^*$.

1.4 Continuity and Differentiability of an Inverse Function

Continuity and Differentiability of an Inverse Function

- If $A \subseteq \mathbb{C}$ is compact and $f : A \to \mathbb{C}$ is continuous and 1-1, then its inverse f^{-1} is continuous.
- If $f : A \to \mathbb{C}$ is 1-1 and differentiable at $z_0 \in A$ and if f^{-1} is continuous at $w_0 = f(z_0)$ (*assumed* to be interior to $f(A)$), then f^{-1} is differentiable at w_0 and

$$\left(f^{-1}\right)'(w_0) = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}$$

- In our definition of differentiability at $z_0 \in A$ we required that z_0 be interior to A. We could have required only that $z_0 \in A$ is not an isolated point of A.
- We are concerned with analytic functions so this extension of the definition will be of no interest.
- We shall see later that an analytic function maps an open set onto an open set: The *open mapping property* of analytic functions (Theorem 5 p. 363). It does not require f to be 1-1.

1.5 The Logarithmic Function II

The Logarithmic Function II

- When $\tau \in \mathbb{R}$ we defined $\arg_\tau(z)$ as that argument to z which lies in the interval $]\tau, \tau + 2\pi]$.
- *Definition*. For $z \in \mathbb{C} \setminus \{0\}$ we define

$$\mathcal{L}_\tau(z) = \ln|z| + i \arg_\tau z$$

- $\mathcal{L}_\tau$ is the inverse of the restriction of exp to the strip given by $\tau < \operatorname{Im} z \leq \tau + 2\pi$.
- $\mathcal{L}_\tau$ is analytic in $D_\tau = \mathbb{C} \setminus \left\{te^{i\tau} \mid t \geq 0\right\}$ and $\frac{d}{dz}\mathcal{L}_\tau(z) = \frac{1}{z}$ for all $z \in D_\tau$.
- A *branch* of a multiple-valued function f in a domain D is a single-valued continuous function F in D such that for each $z \in D$ $F(z)$ is one of the values of $f(z)$.
- Thus $\mathcal{L}_\tau$ is a branch of log in $D_\tau = \mathbb{C} \setminus \left\{te^{i\tau} \mid t \geq 0\right\}$.
- Example. Determine a branch of $f(z) = \log\left(z^3 - 2\right)$ that is analytic at 0.

1.6 Complex Powers I

Complex Powers I

- For $n \in \mathbb{Z}$ and $z \neq 0$ we have $z^n = \left(e^{\log z}\right)^n = e^{n \log z}$.
- *Definition*. For $\alpha \in \mathbb{C}$ and $z \neq 0$ we define

$$z^\alpha = e^{\alpha \log z}$$

- Each branch of the logarithm gives rise to a branch of $z \mapsto z^\alpha$.
- Two values of z^α corresponding to different values of $\log z$ can only be equal if α is real and rational.
- When $m, n \in N$ we get

$$\begin{aligned} z^{\frac{m}{n}} &= \exp\left(\frac{m}{n}\log z\right) = \exp\left(\frac{m}{n}\left(\ln|z| + i\left(Arg(z) + p2\pi\right)\right)\right) \\ &= \exp\left(\frac{m}{n}\ln|z|\right)\exp\left(i\frac{m}{n}Arg(z)\right)\exp\left(i\frac{m}{n}p2\pi\right) \end{aligned}$$

  These are different for $p = 0, 1, \ldots, n-1$ (assuming m and n relatively prime).

- Notice that the values of $z^{\frac{m}{n}}$ are the solutions ζ to the binomial equation $\zeta^n = z^m$.

1.7 Complex Powers II

Complex Powers II

- *Definition*. For $\alpha \in \mathbb{C}$ and $z \neq 0$ *the principal branch* of z^α is

$$z^\alpha = e^{\alpha Log(z)}$$

- We have $\frac{d}{dz}\left(e^{\alpha Log(z)}\right) = e^{\alpha Log(z)} \cdot \frac{\alpha}{z}$ when $z \in \mathbb{C} \setminus \,]-\infty, 0]$.

- We have $\frac{d}{dz}\left(z^\alpha\right) = \alpha z^\alpha \cdot \frac{1}{z}$ when the same branch of z^α is used on both sides.

- Example 2 (§3.5). A branch of $\left(z^2 - 1\right)^{\frac{1}{2}}$ that is analytic outside the unit circle. Use $z\left(1 - \frac{1}{z^2}\right)^{\frac{1}{2}}$. See Maple worksheet.

1.8 Inverse Trigonometric Functions

Inverse Trigonometric Functions

- *Definition*. The inverse sine $\sin^{-1} z$ (or $\arcsin z$) is a multiple-valued function: the solution w to $z = \sin w$.

- We have $\arcsin z = -i \log\left(iz + \left(1 - z^2\right)^{\frac{1}{2}}\right)$.

- For any branch of arcsin we have $\frac{d}{dz} \arcsin z = \frac{1}{(1-z^2)^{\frac{1}{2}}}$.

- Example. When $z \in \,]-1, 1[$ and principal values are used for both *Log* and square root we get $\text{Arcsin}(z) \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[$.

- ± 1 are branchpoints for arcsin.