Complex Analysis
Week 5: §§2.5, 3.4, 7.1-7.2

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October 2, 2008

1 Harmonic Functions and Conformal Mappings

1.1 Harmonic Functions I

Harmonic Functions I

- Let $D \subseteq \mathbb{R}^2$ be open. $C^k(D)$ is the set of real-valued functions having continuous partial derivatives up to and including order k in D. $C^\infty(D)$ is the set of functions belonging to $C^k(D)$ for each $k \in \mathbb{N}$.
- Let $D \subseteq \mathbb{R}^2$ be open and connected (a domain in $\mathbb{R}^2$) and let $\phi \in C^2(D)$. If ϕ satisfies
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$
in all of D, then ϕ is said to be *harmonic* in D.
- When $D \subseteq \mathbb{R}^2$ we may also consider D a subset of $\mathbb{C}$ in the obvious way.
- When $f : D \subseteq \mathbb{C}$ we shall consider its real and imaginary parts u and v as functions of two real variables: $f(x+iy) = u(x,y) + iv(x,y)$.

1.2 Harmonic Functions II

Harmonic Functions II

- Let $D \subseteq \mathbb{C}$ be a domain and let $f = u + iv$ be analytic in D. Then u and v are harmonic in D.
- Proof. It will be shown later in the course that if f is analytic in D then so is its derivative f', a quite remarkable fact and not at all obvious!
- Thus if f is analytic $u, v \in C^\infty(D)$.
- u and v satisfy the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$.
- By the equality of mixed partials it follows that
$$\begin{aligned} u_{xx} &= v_{yx} = v_{xy} = -u_{yy} \\ v_{xx} &= -u_{yx} = -u_{xy} = -v_{yy} \end{aligned}$$

- Conversely: If u is a harmonic function in a *simply connected* domain D then there exists another harmonic function v (a *harmonic conjugate* of u) such that $f = u + iv$ is analytic in D.

1.3 Harmonic Functions III

Harmonic Functions III

- Proof. Let u be harmonic in the simply connected domain D. We shall determine v such that $f = u + iv$ is analytic in D.
- v must satisfy $v_x = -u_y$ and $v_y = u_x$. Thus v is an antiderivative to the differential form $-u_y dx + u_x dy$.
- Using results from *Matematik 1* such an antiderivative exists iff $\frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial y}(-u_y)$.
- But this equation is satisfied since u is harmonic.
- Example A. See Maple worksheet Lweek05.mw or Example 1 (p.80) in Saff&Snider.
- When u and v are harmonic conjugates: $\nabla u \cdot \nabla v = u_x v_x + u_y v_y = u_x(-u_y) + u_y u_x = 0$.
- Thus the level curves of u and v meet at right angles if the gradients are nonzero at the intersection. Maple example.

1.4 Harmonic Functions IV: Dirichlet problem in annular domain

Harmonic Functions IV: Dirichlet problem in annular domain

- Find a function harmonic between the circles $(x-2)^2 + (y-2)^2 = 1$ and $(x-2)^2 + (y-2)^2 = 4$ and on these circles having the constant boundary values 1 and 0, respectively.
- $\phi(x,y) = \mathrm{Re}(Log(x + iy - 2 - 2i)) = \frac{1}{2}\ln\left((x-2)^2 + (y-2)^2\right)$ is harmonic in $\mathbb{R}^2 \setminus \{(2,2)\}$. *Not* excluding any more!
- That function is constant on the circles, but does not have the right constant values.
- Consider instead $\phi(x,y) = \frac{1}{2} A \ln\left((x-2)^2 + (y-2)^2\right) + B$.
- Boundary conditions: $A \ln 1 + B = 1$ and $A \ln 2 + B = 0$ so $B = 1$ and $A = \frac{-1}{\ln 2}$.
- Solution: $\phi(x,y) = \frac{-1}{2\ln 2} \ln\left((x-2)^2 + (y-2)^2\right) + 1$. See Maple worksheet Lweek05.mw.

1.5 Harmonic Functions V: Dirichlet problem in a wedge

Harmonic Functions V: Dirichlet problem in a wedge

- Find a function harmonic in the wedge-shaped sector given by $z = -1 + re^{i\theta}, r \geq 0, \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ and on two boundary lines takes the values 7 and 3, respectively.
- $\phi(x, y) = Arg(x + 1 + iy) = \arg_{-\pi}(x + 1 + iy)$ is harmonic in the given sector since $\text{Im}(Log(x + 1 + iy)) = Arg(x + 1 + iy)$.
- That function is constant on the half-lines, but does not have the right constant values.
- Consider instead $\phi(x, y) = A \cdot Arg(x + iy) + B$.
- Boundary conditions: $A\frac{\pi}{4} + B = 7$ and $A\frac{3\pi}{4} + B = 3$ so $A = -\frac{8}{\pi}$ and $B = 9$.
- Solution: $\phi(x, y) = \frac{-8}{\pi} Arg(x + 1 + iy) + 9$. See Maple worksheet Lweek05.mw.

1.6 Conformal Mapping I: Invariance of $\Delta\phi = 0$ (§7.1)

Conformal Mapping I: Invariance of Laplace's Equation (§7.1)

- Last time: If D is a domain and $f : D \to \mathbb{C}$ is 1-1 and analytic, (and if $f(D)$ is open and $f'(z) \neq 0$), f^{-1} is analytic, and $\left(f^{-1}\right)'(w_0) = \frac{1}{f'(z_0)} = \frac{1}{f'\left(f^{-1}(w_0)\right)}$.
- Let f be analytic and 1-1 from the domain D onto $D' = f(D)$. We write $u + iv = w = f(z) = f(x + iy)$.
- Let ϕ be harmonic in D. Then $\psi(u, v) = \phi\left(f^{-1}(u + iv)\right)$ is harmonic in $D' = f(D)$.
- Proof. In each open disk $B \subseteq D$ ϕ is the real part of an analytic function g (possibly depending on B). Thus in $f(B)$ the function $g \circ f^{-1}$ is analytic. Thus $\psi(u, v) = \text{Re}\, g\left(f^{-1}(u + iv)\right)$ is harmonic in $f(B)$.
- $D = \bigcup_i B_i$ (B_i's open disks) so $D' = f(D) = \bigcup_i f(B_i)$. Thus ψ is harmonic in all of $f(D)$.

1.7 Conformal Mapping II: Example 1 (§7.1)

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- Find ϕ harmonic in the open unit disk B and satisfying the boundary condition $\phi = 1$ on the upper boundary semi-circle and $\phi = -1$ on the lower.
- Let $f(z) = \frac{1+z}{1-z}, z \neq 1$. $f(B)$ is the open right half-plane.

- Furthermore

$$f\left(e^{i\theta}\right)=\frac{1+e^{i\theta}}{1-e^{i\theta}}=\frac{e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}}=i\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=i\cot\frac{\theta}{2}$$

- Thus f maps the unit circle (excluding 1) onto the imaginary axis. The upper (lower) half-circle is mapped onto the positive (negative) imaginary axis.

- $\psi(u,v)=A\cdot Arg(w)+B$ is harmonic in the right half-plane. Using the boundary conditions we find $A=\frac{2}{\pi}, B=0$.

- Thus $\phi(x,y)=\psi(f(z))=\frac{2}{\pi}Arg\left(\frac{1+z}{1-z}\right)=\frac{2}{\pi}\arctan\frac{2y}{1-x^2-y^2}$. See Maple worksheet Lweek05.mw.

1.8 Conformal Mapping III: Theorem 1 (§7.2)

Conformal Mapping III: Theorem 1 (§7.2)

- Theorem 1. If f is analytic at z_0 and $f'(z_0)\neq 0$ then there exists an open disk D centered at z_0 such that f is 1-1 on D.

- Proof using definition of analyticity and continuity of derivative.

- May assume that $|f'(z)|\geq a>0$ for all $z\in D$ by picking a smaller disk if necessary.

- For any $z_1\in D$ there exists $\delta>0$ such that if $|z_1-z_2|<\delta\wedge z_2\in D$ then

$$\left|f(z_2)-f(z_1)-f'(z_1)(z_2-z_1)\right|<\frac{1}{2}a|z_1-z_2|$$

- We can also pick a $\delta>0$ common to all $z_1\in D$.

- By the inverse triangle inequality $|f(z_2)-f(z_1)|$ is greater than or equal to

$$\begin{aligned}&\left|f'(z_1)(z_2-z_1)\right|-\left|f(z_2)-f(z_1)-f'(z_1)(z_2-z_1)\right|\\ \geq\ & a|z_1-z_2|-\frac{1}{2}a|z_1-z_2|=\frac{1}{2}a|z_1-z_2|\end{aligned}$$

1.9 Conformal Mapping IV: Theorem 2 (§7.2)

Conformal Mapping IV: Theorem 2 (§7.2)

- Definition of conformality: Angle preservation including orientation!

- Theorem 2. If f is analytic in D and if in $z_0\in D$ we have $f'(z_0)\neq 0$ then f is conformal at z_0.

- Proof. Let γ be a curve given by a differentiable parametrization $z=z(t)$ with $z(t_0)=z_0$.

- Then $w(t) = f(z(t))$ is a parametrization of the image of the curve γ.
- We find
$$w'(t_0) = f'(z(t_0))\, z'(t_0) = f'(z_0)\, z'(t_0)$$
- But multiplication by $f'(z_0) = re^{i\theta}$ corresponds to a rotion by θ (and a scaling by $r > 0$).
- Both tangents to any two curves through z_0 are rotated the same amount, so the angle between the two is preserved.

1.10 Conformal Mapping V: Theorems 3 and 4 (§7.2)

Conformal Mapping V: Theorems 3 and 4 (§7.2)

- Theorem. *Conformality implies analyticity*: Suppose $f(z) = u(x,y) + iv(x,y)$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) with a regular Jacobian matrix. Suppose further that f is conformal at (x_0, y_0). Then $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 = x_0 + iy_0$.
- Proof: We only need show that conformality implies that the Cauchy-Riemann equations are satisfied at the point (x_0, y_0). Not difficult, see e.g. Nehari, Conformal Mapping, pp. 150-152. The computation is also shown in the Maple worksheet Lweek05.mw.
- Theorem 3. The Open Mapping Theorem. If f is analytic and nonconstant in the domain D then f maps open subsets of D onto open sets.
- Theorem 4. The Riemann Mapping Theorem. Let $D \neq$C be a simply connected domain. Then there exists a 1-1 analytic function f that maps D onto the open unit disk. With $z_0 \in D$ the mapping is uniquely defined if we require $f(z_0) = 0$ and $f'(z_0) > 0$.