Week 6: §§7.3 -7.4

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1 Conformal Mapping: Möbius Transformations

1.1 Möbius Transformation: Definition

Möbius Transformation: Definition

- Let f be defined by
$$f(z) = \frac{az+b}{cz+d}$$
where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$. Then f is called a *Möbius transformation* or a *fractional linear transformation*.

- $ad - bc \neq 0$ implies that the denominator is not identically zero and that f is not a constant function, since
$$f'(z) = \frac{ad-bc}{(cz+d)^2}$$

- f is 1-1 on its domain of definition and the inverse f^{-1} is again a Möbius transformation and is given by
$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

1.2 Möbius Transformation: Matrices

Möbius Transformation: Matrices

- Let the Möbius transformations f and g be given by
$$f(z) = \frac{az+b}{cz+d},\ g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

- Associate with f and g the coefficient matrices $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $G = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

- The composition of the Möbius transformations f and g is again a Möbius transformation, and $f \circ g$ has the coefficient matrix FG.

- f^{-1} has the coefficient matrix F^{-1} given by

$$F^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- But we may as well use the matrix obtained by removing the factor $(ad-bc)^{-1}$.

1.3 Möbius Transformation: Special cases I

Möbius Transformation: Special cases I

- *Translation* by b: $f(z) = z + b$.
- *Rotation* by θ: $f(z) = e^{i\theta}z$.
- Rotation by θ followed by a *magnification (scaling)* by the factor r: $f(z) = re^{i\theta}z$.
- Rotation followed by a magnification followed by a translation: $f(z) = az + b$.
- Example 2. A linear transformation $f(z) = az + b$ that maps the circle C_1: $|z-1| = 1$ onto the circle C_2: $\left|w - \frac{3}{2}i\right| = 2$.
- Maple: First translate to 0, then magnify by 2, then translate to $\frac{3}{2}i$: $f(z) = 2(z-1) + \frac{3}{2}i$.
- The above-mentioned special Möbius transformations map lines onto lines and circles onto circles. They are all 1-1 on all of $\mathbb{C}$.
- In the extended complex plane they satisfy $f(\infty) = \infty$.

1.4 Möbius Transformation: Special cases II, Inversion

Möbius Transformation: Special cases II, Inversion

- The *inversion* transformation $f(z) = \frac{1}{z}$.
- Defining $f(0) = \infty$ and $f(\infty) = 0$ makes f a 1-1 mapping of the extended complex plane onto itself.
- Let S be the stereographic projection. In Example 4, p. 55 we saw that SfS^{-1} is a rotation by 180° about the x_1-axis. Thus f maps generalized circles onto generalized circles.
- See Maple illustration, where it is also shown that

$$\left(S^{-1}R_tS\right)(z) = \frac{z\cos\frac{t}{2} + i\sin\frac{t}{2}}{zi\sin\frac{t}{2} + \cos\frac{t}{2}}$$

  where R_t is rotation by an angle of t about the x_1-axis.
- If Σ_+ and Σ_- are the upper and lower Riemann hemispheres, then $SfS^{-1}\Sigma_- = \Sigma_+$.

1.5 Möbius Transformation: General properties

Möbius Transformation: General properties

- Let $f(z) = \frac{az+b}{cz+d}$. Suppose $c \neq 0$. Then f can be written

$$f(z) = -\frac{1}{c}\frac{ad-bc}{cz+d} + \frac{a}{c}$$

- Thus when $c \neq 0$ then f can be expressed as a linear transformation L_1 followed by an inversion I followed by a linear transformation L_2: $f = L_2 \circ I \circ L_1$.
- Written differently $f(z) = L_2\left(\frac{1}{L_1(z)}\right)$.
- Define $f\left(-\frac{d}{c}\right) = \infty$ and $f(\infty) = \frac{a}{c}$. Then f is a 1-1 mapping of the extended complex plane onto itself.
- f maps generalized circles onto generalized circles.
- Since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ the Möbius transformation f is conformal everywhere except at $-\frac{d}{c}$.

1.6 Möbius Transformation: Example A

Möbius Transformation: Example A

- Let f be given by

$$f(z) = \frac{1+3i-z}{z-1-i}$$

- We find the image of the interior of the circle C given by $|z-1-2i| = 2$.
- Since $f(1+i) = \infty$ and $1+i \notin C$ we see that $f(C)$ is a circle. Since $f(1) = -3$ and $f(1+4i) = -\frac{1}{3}$ and since C intersects the line connecting 1 and $1+4i$ at right angles, by conformality $f(C)$ is the circle having the line segment $\left[-3, -\frac{1}{3}\right]$ as a diameter.
- Since the interior and the exterior of C both are connected, so are their images under f.
- Since $1+i$ belongs to the interior of C and is mapped to ∞ it follows that the image of the interior of C is the exterior of the circle $\left|z+\frac{5}{3}\right| = \frac{4}{3}$.

1.7 Möbius Transformation: Example B

Möbius Transformation: Example B

- Let f still be given by

$$f(z) = \frac{1+3i-z}{z-1-i}$$

- Now find the image of the interior of the (new) circle C given by $|z - 1 - 2i| = 1$.
- Since $f(1 + i) = \infty$ and $1 + i \in C$ we see that $f(C)$ is a line. Since $f(2i) = -i$ and $f(1 + 3i) = 0$ the line must be the imaginary axis.
- Since the interior and the exterior of C both are connected, so are their images under f.
- Since $f(1 + 2i) = 1$ lies in the right half-plane, the interior of C is mapped onto the right half-plane.

1.8 Möbius Transformation: Example 4 (§7.3)

Möbius Transformation: Example 4 (§7.3)

- Find a conformal map of the unit disk $|z| < 1$ onto the right half-plane.
- Look for a Möbius transformation f mapping the unit circle onto the imaginary axis.
- If $f(0)$ happens to lie in the left half-plane we use $-f$ instead.
- Pick an arbitrary point z_1 on the circle and require $f(z_1) = \infty$.
- Pick another two arbitrary points z_2 and z_3 on the circle and require e.g. $f(z_2) = i$ and $f(z_3) = 0$.
- $f(z_1) = \infty$ and $f(z_3) = 0$ imply that $f(z) = A\frac{z-z_3}{z-z_1}$.
- $f(z_2) = i$ implies that $A\frac{z_2-z_3}{z_2-z_1} = i$.
- In the book $z_1 = 1, z_3 = -1$ and $z_2 = -i$ and we first find $f(z) = \frac{z+1}{z-1}$. But $f(0) = -1$ so take instead $f(z) = -\frac{z+1}{z-1}$.

1.9 Möbius Transformation: The Cross-Ratio (§7.4)

Möbius Transformation: The Cross-Ratio (§7.4)

- Pick 3 arbitrary, but different points z_1, z_2, z_3 in the extended complex plane and require $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$.
- Suppose first that all 3 points are finite.
- $f(z_1) = 0$ and $f(z_3) = \infty$ imply that

$$f(z) = A\frac{z - z_1}{z - z_3}$$

- $f(z_2) = 1$ implies that

$$A\frac{z_2 - z_1}{z_2 - z_3} = 1$$

- Thus f is given by the *cross-ratio*

$$f(z) = (z, z_1, z_2, z_3) := \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

- If one of the 3 points z_1, z_2, z_3 is ∞ then the cross-ratio is found by simply removing the factors in the numerator and denominator involving ∞.

1.10 Möbius Transformation a la carte (§7.4)

Möbius Transformation a la carte (§7.4)

- Pick 3 arbitrary, but different points z_1, z_2, z_3 in the extended complex z-plane and 3 arbitrary but different points w_1, w_2, w_3 inthe extended complex w-plane.
- Require $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3$.
- Let $T(z) = (z, z_1, z_2, z_3)$ and let $S(w) = (w, w_1, w_2, w_3)$.
- Then $f = S^{-1} \circ T$ satisfies the requirements.
- This means that $w = f(z) = S^{-1}(T(z))$. Thus $S(w) = T(z)$.
- This means that we can find $w = f(z)$ by solving $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$ for w.
- Orientation: Since a Möbius transformation is conformal it maps the region *left* of the generalized circle C onto the region *left* of the generalized circle $f(C)$.
- Examples 1 and 2 can be found in the Maple worksheet for Chapter 7.