Week 7: §§4.1-4.3

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1 Complex Integration

1.1 Smooth Curves: Definitions

Smooth Curves: Definitions

- γ is a *smooth arc* (Matematik 1: *en differentiabel kurve*) if it is given by a 1-1 C^1-parametrization $\phi : I = [a, b] \to \mathbb{C}$, satisfying $\phi'(t) \neq 0$ for all $t \in I$.
- γ is a *smooth closed curve* if it is given by a C^1-parametrization $\phi : I = [a, b] \to \mathbb{C}$ being 1-1 on $[a, b[$ and satisfying $\phi'(t) \neq 0$ for all $t \in [a, b[$, but $\phi(b) = \phi(a)$ and $\phi'(b) = \phi'(a)$.
- A parametrization that satisfies the respective requirements is called *admissible*.
- In a *directed smooth arc* or a *directed smooth closed* curve we include the orientation.
- A (*directed*) *smooth curve* is the common term for a (directed) smooth arc and a (directed) smooth closed curve.
- If $\phi : [a, b] \to \mathbb{C}$ parametrizes γ then $\psi(t) = \phi(a + b - t), t \in [a, b]$, parametrizes $-\gamma$.

1.2 Contours: Definitions

Contours: Definitions

- A contour Γ is either a single point or a finite sequence of directed smooth curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ where the terminal point of γ_k coincides with the initial point of γ_{k+1} for $k = 1, 2, \ldots, n - 1$.
- We may write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.
- When we disregard the direction of Γ we shall call it a *piecewise smooth curve*.
- A *closed contour* (or a *loop*) is a contour for which the initial point and the terminal point coincide.
- A *simple closed contour* has no multiple points other than the initial point and the terminal point.

- *The Jordan Curve Theorem*: A simple closed contour Γ separates $\mathbb{C} \setminus \Gamma$ into two domains each having Γ as its boundary. One is bounded and simply connected (the *interior*), the other is unbounded (the *exterior*).
- Maple §4.1.

1.3 Contour Integrals: Definition I

Contour Integral: Definition I

- Matematik 1: If γ is a piecewise C^1-curve having parametrization $r : [a, b] \to R^k$ where r is 1-1 almost everywhere then the length of γ is

$$\ell(\gamma) = \int_a^b \|r'(t)\| \, dt$$

- In our context: If Γ is a contour having piecewise C^1-parametrization $\phi : [a, b] \to \mathbb{C}$ then

$$\ell(\Gamma) = \int_a^b |\phi'(t)| \, dt$$

- Let $\mathcal{P}_n = z_0 = \alpha, z_1, \ldots z_{n-1}, z_n = \beta$ be a partition of a directed smooth curve γ having initial point α and final point β.
- Choose for each k a point c_k on γ between z_{k-1} and z_k. Let $c = (c_k)_{k=1}^n$. c is *associated* with $\mathcal{P}_n$. The *mesh* of the partition $\mathcal{P}_n$ is defined by

$$\mu(\mathcal{P}_n) = \max_k |z_k - z_{k-1}|$$

1.4 Contour Integrals: Definition II

Contour Integral: Definition II

- The *Riemann sum* $S(\mathcal{P}_n, c)$ for the function f is defined as

$$S(\mathcal{P}_n, c) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

- f is *integrable along* γ if there exist a complex number L such that

$$\begin{aligned} \forall\varepsilon &> 0 \; \exists\delta > 0 \text{ s.t.} \\ \mu(\mathcal{P}_n) &< \delta \Rightarrow |L - S(\mathcal{P}_n, c)| < \varepsilon \text{ for all } c \text{ ass. with } \mathcal{P}_n \end{aligned}$$

- The number L is uniquely determined if it exists and is called the integral of f along γ:

$$L = \int_\gamma f(z) \, dz$$

1.5 Contour Integrals: Theorems 2 and 4

Contour Integral: Theorems 2 and 4

- Theorem 2. If f is continuous on the directed smooth curve γ, then f is integrable along γ.
- Theorem 4. Let f be continuous on the directed smooth curve γ and let $z = z(t), t \in [a, b]$, be an admissible parametrization consistent with the direction of γ. Then

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

- Definition 4. Let $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ be a contour and f a function continuous on Γ. Then we define

$$\int_\Gamma f(z)\, dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\, dz$$

- It is debatable whether this should really have been a theorem since the definition of integral along a curve might as well have included contours to begin with.

1.6 Contour Integrals: Theorem 5

Contour Integral: Theorem 5

- Maple §4.2 Example 2, 3, and one similar to 4.
- Theorem 5. If f is continuous on the contour Γ and if $|f(z)| \leq M$ for all $z \in \Gamma$ then

$$\left| \int_\Gamma f(z)\, dz \right| \leq M\, \ell(\Gamma)$$

- Proof. We need only consider a smooth curve γ with admissible parametrization $z = z(t), t \in [a, b]$. Then

$$\begin{aligned} \left| \int_\gamma f(z)\, dz \right| &= \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leq \int_a^b \left| f(z(t))\, z'(t) \right| dt \\ &\leq M \int_a^b \left| z'(t) \right| dt = M\, \ell(\Gamma) \end{aligned}$$

- Maple §4.2 Example "5". $f(z) = \frac{Log(z)}{z^4+1}$ where $\Gamma : 2e^{it}, t \in [0, \pi]$. Bound: $2\pi \frac{\sqrt{(\ln 2)^2 + \pi^2}}{15}$.

1.7 Contour Integrals: The square root

Contour Integral: The square root

- Let $f(z) = z^{\frac{1}{2}} = \exp\left(\frac{1}{2}\log_\tau(z)\right)$ where $\tau \in \mathbb{R}$.
- We find the integral $\int_\Gamma f(z)\,dz$ where Γ is the unit circle (counter-clockwise).
- A suitable parametrization is $t \to e^{it}, t \in [\tau, \tau + 2\pi]$.
- $\int_\Gamma f(z)\,dz = \int_\tau^{2\pi+\tau} f\left(e^{it}\right) ie^{it}dt = \int_\tau^{2\pi+\tau} \exp\left(\frac{1}{2}\log_\tau\left(e^{it}\right)\right) ie^{it}dt$
- Since $\log_\tau\left(e^{it}\right) = \ln\left|e^{it}\right| + i\arg_\tau\left(e^{it}\right) = it$ when $t \in [\tau, \tau + 2\pi]$ we get
- $\int_\Gamma f(z)\,dz = \int_\tau^{2\pi+\tau} e^{\frac{1}{2}it} ie^{it}dt = i\int_\tau^{2\pi+\tau} e^{\frac{3}{2}it}dt = \frac{2}{3}\left[e^{\frac{3}{2}it}\right]_\tau^{2\pi+\tau} = \frac{2}{3}\left(e^{\frac{3}{2}i(2\pi+\tau)} - e^{\frac{3}{2}i\tau}\right) = -\frac{4}{3}e^{\frac{3}{2}i\tau}$.
- We notice that the result is dependent on the branch!
- Exercise: Try the same for $f(z) = \log_\tau(z)$. (Answer: $2\pi e^{i\tau}$).

1.8 Independence of Path I

Independence of Path I

- Theorem 6. Suppose f is continuous in a domain D and has an antiderivative F in D. Let Γ be a contour in D having initial point z_I and terminal point z_T. Then
$$\int_\Gamma f(z)\,dz = F(z_T) - F(z_I)$$
- Proof. We need only consider a smooth curve γ with admissible parametrization $z = z'(t), t \in [a, b]$. Then
$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b F'(z(t))\,z'(t)\,dt \\ &= \int_a^b \frac{d}{dt}\left(F(z(t))\right)dt = F(z(b)) - F(z(a)) \\ &= F(z_T) - F(z_I) \end{aligned}$$
- Corollary 2. If f is continuous in the domain D and has an antiderivative in D, then $\int_\Gamma f(z)\,dz = 0$ for all loops Γ in D.

1.9 Independence of Path II

Independence of Path II

- Maple §4.3 Example 1 and 2.
- Theorem 7. The Equivalence Theorem. Let f be continuous in a domain D. Then the following three statements are equivalent:

1. f has an antiderivative in D.
2. Every loop integral of f in D vanishes, i.e. $\int_\Gamma f(z)\,dz = 0$ for any loop Γ in D.
3. The contour integrals of f are independent of path in D.

1.10 Independence of Path III

Independence of Path III

- Proof. We prove that (3) implies (1). Thus assume that contour integrals of f are independent of path in D.
- We shall show that f has an antiderivative in D.
- Define F by $F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$ where the path connecting $z_0 \in D$ with $z \in D$ lies entirely in D.
- For small values of $h \in \mathbb{C}$ the points $z + th \in D$ for all $t \in [0,1]$. Thus $F(z+h) - F(z) = \int_{z}^{z+h} f(\zeta)\,d\zeta = \int_0^1 f(z+th)\,hdt$.
- By the continuity of f at z we find

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+th)\,dt \to f(z) \text{ as } h \to 0$$

- Thus F is differentiable at z and $F'(z) = f(z)$.