Week 8: §§4.4a-4.5
Cauchy's Integral Theorem and Formula

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1 Cauchy-Goursat Theorem for a Disk

Cauchy-Goursat Theorem for a Disk

- Theorem 9. If f is analytic in the open disk D then f has an antiderivative in D and for any closed contour Γ in D

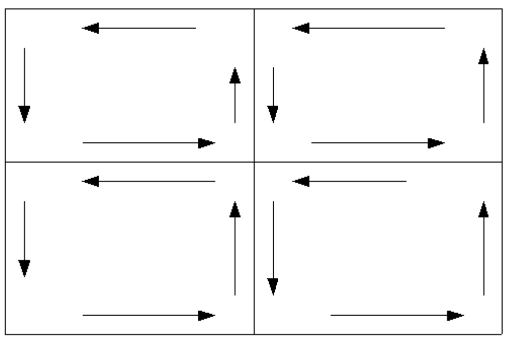
$$\int_\Gamma f(z)\,dz = 0$$

- The proof is split into three parts:
- Lemma 1. If $R \subset D$ is a rectangle with sides parallel to the axes then $\int_R f(z)\,dz = 0$.
- Lemma 2. If $\int_R f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes then f has an antiderivative in D.
- By the Equivalence Theorem we conclude that loop integrals of f in D are zero.

1.1 Proof of Lemma 1, I

Proof of Lemma 1, I

- Lemma 1. If $R \subset D$ is a rectangle with sides parallel to the axes then $\int_R f(z)\,dz = 0$.



- $\int_R f(z)\,dz = \int_{R_1} f(z)\,dz + \int_{R_2} f(z)\,dz + \int_{R_3} f(z)\,dz + \int_{R_4} f(z)\,dz.$

1.2 Proof of Lemma 1, II

Proof of Lemma 1, II

- Since $\left|\int_R f(z)\,dz\right| \le \left|\int_{R_1} f(z)\,dz\right| + \left|\int_{R_2} f(z)\,dz\right| + \left|\int_{R_3} f(z)\,dz\right| + \left|\int_{R_4} f(z)\,dz\right|$ at least one of the subrectangles R_k satisfies

$$\left|\int_{R_k} f(z)\,dz\right| \ge \frac{1}{4}\left|\int_R f(z)\,dz\right|$$

- Select such a subrectangle, name it $R^{(1)}$, and divide it in four pieces as above: $R_k^{(1)}, k = 1,2,3,4$. Then for (at least) one of those (named $R^{(2)}$) we have

$$\left|\int_{R^{(2)}} f(z)\,dz\right| \ge \frac{1}{4}\left|\int_{R^{(1)}} f(z)\,dz\right| \ge \frac{1}{4^2}\left|\int_R f(z)\,dz\right|$$

- Continuing in this way we obtain a sequence of rectangles $R, R^{(1)}, R^{(2)}, \ldots, R^{(n)}, \ldots$ satisfying

$$\left|\int_{R^{(n)}} f(z)\,dz\right| \ge \frac{1}{4^n}\left|\int_R f(z)\,dz\right|$$

1.3 Proof of Lemma 1, III

Proof of Lemma 1, III

- If P and d are the perimeter and diagonal of R, respectively, then the perimeter and diagonal of $R^{(n)}$ are $2^{-n}P$ and $2^{-n}d$, respectively.
- The sequence of upper left hand corners of the rectangles must converge to a point z_0, which belongs to all the rectangles.
- If $z \in R^{(n)}$ then $|z - z_0| \le 2^{-n}d$.
- Since f is differentiable at z_0 there exists for given $\varepsilon > 0$ a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$$

- Pick n such that $2^{-n}d \le \delta$. Then for $z \in R^{(n)}$ we have

$$\left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| < \varepsilon\,|z - z_0| \le \varepsilon 2^{-n} d$$

1.4 Proof of Lemma 1, IV

Proof of Lemma 1, IV

- Since $\int_{R^{(n)}} 1 dz = 0$ and $\int_{R^{(n)}} (z - z_0)\, dz = 0$ we get

$$\begin{aligned}\left|\int_{R^{(n)}} f(z)\, dz\right| &= \left|\int_{R^{(n)}} \left(f(z) - f(z_0) - f'(z_0)(z - z_0)\right) dz\right| \\ &\leq \max_{z \in R^{(n)}} \left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \ell\left(R^{(n)}\right) \\ &\leq \varepsilon d 2^{-n} P 2^{-n} = \varepsilon d P 4^{-n}\end{aligned}$$

- Thus

$$\left|\int_{R} f(z)\, dz\right| \leq 4^n \left|\int_{R^{(n)}} f(z)\, dz\right| \leq \varepsilon d P$$

- This inequality holds for every $\varepsilon > 0$ so $\int_R f(z)\, dz = 0$.

1.5 Proof of Lemma 2, I

Proof of Lemma 2, I

- Lemma 2. Let D be an open disk. If $\int_R f(z)\, dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes then f has an antiderivative in D.
- For any $z_1, z_2 \in D$ let $hv(z_1, z_2)$ be the path from z_1 to z_2 composed of first a horizontal line segment then a vertical line segment.
- Let z_0 be the center of D. Define F by

$$F(z) = \int_{hv(z_0, z)} f(\zeta)\, d\zeta$$

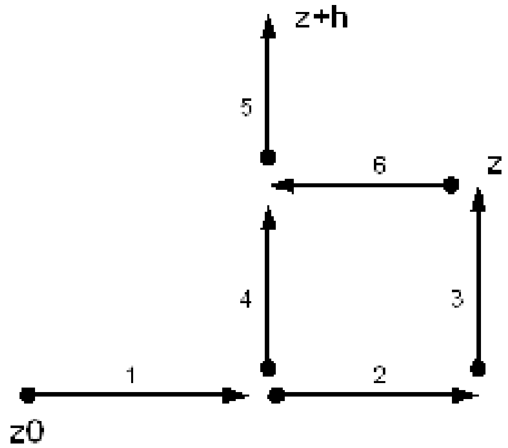
- We shall show that $F'(z) = f(z)$ for any $z \in D$, which means showing that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

1.6 Proof of Lemma 2, II

Proof of Lemma 2, II

- $F(z+h) - F(z) = \left(\int_{\gamma_1 + \gamma_4 + \gamma_5} - \int_{\gamma_1 + \gamma_2 + \gamma_3}\right) f(\zeta)\, d\zeta.$



- $\int_{\gamma_2+\gamma_3+\gamma_6-\gamma_4} f(\zeta)\, d\zeta = 0$ so $F(z+h) - F(z) = \int_{\gamma_5+\gamma_6} f(\zeta)\, d\zeta = \int_{hv(z,z+h)} f(\zeta)\, d\zeta$.

1.7 Proof of Lemma 2, III

Proof of Lemma 2, III

- Since $F(z+h) - F(z) = \int_{hv(z,z+h)} f(\zeta)\, d\zeta$ we get
$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{hv(z,z+h)} (f(\zeta) - f(z))\, d\zeta$$
- For $\varepsilon > 0$ given we determine $\delta > 0$ s.t. $|\zeta - z| < \delta$ implies $|f(\zeta) - f(z)| < \varepsilon$. Now for $\zeta \in hv(z, z+h)$ $|\zeta - z| < 2|h|$. Thus for $2|h| < \delta$
$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \varepsilon \ell (hv(z, z+h)) \leq 2\varepsilon$$
- This completes the proof of Lemma 2 and the Cauchy-Goursat Theorem for a Disk.

2 Cauchy's Integral Formula for Circle in Disk I

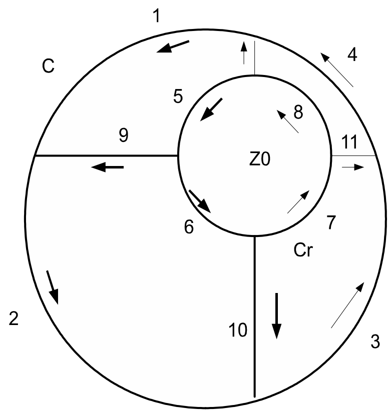
Cauchy's Integral Formula for Circle in Disk I

- Theorem. Let f be analytic in the open disk D. Let C be a circle contained in a square inscribed in D (traversed counterclockwise). Then for any z_0 inside C
$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$
- The requirement that C be inside a square inscribed in D makes the proof easier.
- We get a more general version later anyway.
- Proof. Let C_r be the positively oriented circle with center z_0 and radius $r > 0$ small enough to ensure that C_r is inside C.

2.1 Proof I

Proof I

- $\frac{f(z)}{z-z_0}$ is analytic in each of 4 disks containing a piece of pineapple. Integrals around them are zero. So the sum of these is zero. Contributions from straight line segments cancel.



2.2 Proof II

Proof II

- Thus $\int_C \frac{f(z)}{z-z_0}dz = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} = \int_{\gamma_5+\gamma_6+\gamma_7+\gamma_8} = \int_{C_r} \frac{f(z)}{z-z_0}dz.$
- So $\int_C \frac{f(z)}{z-z_0}dz = \int_{C_r} \frac{f(z)}{z-z_0}dz = \int_{C_r} \frac{f(z_0)}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz.$
- It follows from the equation above that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz$ is independent of r.
- We shall show that the value is zero.

2.3 Proof III

Proof III

- Since f is differentiable at z_0 we have $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$ as $z \to z_0$ and as a consequence there exist an $r_0 > 0$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| \le |f'(z_0)| + 1$$

as long as $|z - z_0| \leq r_0$.

- Thus for $r \leq r_0$ we have

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \left(\left| f'(z_0) \right| + 1 \right) 2\pi r$$

- Being independent of r it follows that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz = 0$.

3 A Differentiability Lemma

A Differentiability Lemma

- Theorem 15. Let g be continuous on the contour Γ. Define

$$G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

for each z not on Γ. Then G is analytic at each z not on Γ and

$$G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$$

- Theorem 15 generalized. Let g be continuous on the contour Γ. Define G as above. Then $G^{(k)}$ is analytic for all $k \geq 0$ at each z not on Γ and

$$G^{(k)}(z) = k! \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

3.1 Proof

Proof

- We have

$$\frac{G(z+h) - G(z)}{h} = \int_\Gamma \frac{g(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta$$

- But

$$\frac{1}{(\zeta - z - h)(\zeta - z)} - \frac{1}{(\zeta - z)^2} = \frac{h}{(\zeta - z - h)(\zeta - z)^2}$$

- Let M be an upper bound for $|g|$ on Γ and $d > 0$ a lower bound for the distance from z to Γ. For $|h| \leq \frac{1}{2}d$ we have $|\zeta - z - h| \geq |\zeta - z| - |h| \geq \frac{1}{2}d$ so

$$\left| \frac{g(\zeta)}{(\zeta - z - h)(\zeta - z)^2} \right| \leq \frac{M}{\frac{1}{2}d^3}$$

- Thus

$$\left| \frac{G(z+h) - G(z)}{h} - \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq |h| \frac{2M}{d^3} \ell(\Gamma)$$

3.2 Theorem 16

Theorem 16

- If f is analytic in the open disk D and $C \subset D$ is a positively oriented circle then for any z inside C

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

- Thus from Theorem 15 (generalized) it follows that f' is analytic inside C and is given by

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

- Theorem 16. Let f be analytic in the domain D. Then derivatives of all orders exist and are analytic in D.

4 Continuous Deformation

Continuous Deformation

- Definition. The loop Γ_0 is *continuously deformable* to the loop Γ_1 *in the domain D* if there exists a continuous function $h : [0,1] \times [0,1] \to D$ satisfying

  1. $t \mapsto h(s,t)$ ($t \in [0,1]$) is a parametrization of a loop in D for each fixed $s \in [0,1]$.
  2. $t \mapsto h(0,t)$ ($t \in [0,1]$) is a parametrization of Γ_0.
  3. $t \mapsto h(1,t)$ ($t \in [0,1]$) is a parametrization of Γ_1.

- Γ_0 and Γ_1 are also instead said to be *homotopic* in D.
- See Maple examples for illustrations.
- Definition. If every loop in a domain D is homotopic to a point in D then D is said to be *simply connected*.

4.1 Deformation Invariance Theorem

Deformation Invariance Theorem

- Theorem 8. Let f be analytic in the domain D. Then

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz$$

  for any two loops Γ_0 and Γ_1 that are homotopic in D.

- Proof under the assumption that: $h : [0,1] \times [0,1] \to D$ is C^2 in the unit square D. We have proved that f' is analytic, thus continuous.

- $I(s) = \int_{\Gamma_s} f(z)\,dz = \int_0^1 f(h(s,t))\,h_t(s,t)\,dt$. Show that $I(s)$ is constant.
- $I'(s) = \int_0^1 \frac{\partial}{\partial s}(f(h)\,h_t)\,dt = \int_0^1 (f'(h)\,h_s h_t + f(h)\,h_{ts})\,dt$
- $\frac{\partial}{\partial t}(f(h)\,h_s) = f'(h)\,h_s h_t + f(h)\,h_{st} = f'(h)\,h_s h_t + f(h)\,h_{ts}$
- $I'(s) = \int_0^1 \frac{\partial}{\partial t}(f(h)\,h_s)\,dt = f(h(s,1))\,h_s(s,1) - f(h(s,0))\,h_s(s,0) = 0$ for all s.

5 Cauchy's Integral Theorem

Cauchy's Integral Theorem

- Theorem 9. If f is analytic in the simply connected domain D and Γ is a loop in D then
$$\int_\Gamma f(z)\,dz = 0$$
- Proof. Γ is homotopic to a point in D. The integral of f "along a point" is zero.
- Theorem 10. In a simply connected domain an analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals vanish.
- Example. Let Γ be any simple closed curve traversed in the counter-clockwise direction and having 0 in its interior. Then
$$\int_\Gamma \frac{1}{z}dz = \int_{\text{Unit circle}} \frac{1}{z}dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}dt = 2\pi i$$
- Example. From Cauchy's theorem it follows that
$$\int_{|z|=2} \frac{\cos z}{(z-7)(z+3i)}dz = 0$$

5.1 Cauchy's Integral Theorem: Examples

Cauchy's Integral Theorem: Examples

- Example. Let Γ be any circle not passing through a. Then
$$\int_\Gamma \frac{1}{z-a}dz = \begin{cases} 0 & \text{when } a \text{ is outside } \Gamma \\ 2\pi i & \text{when } a \text{ is inside } \Gamma \end{cases}$$
- Example. Let Γ be a simple closed contour going counter-clockwise around the points $0, 1, i$. Then since
$$\frac{1+i}{z(z-1)(z-i)} = \frac{1-i}{z} + \frac{i}{z-1} - \frac{1}{z-i}$$

we find

$$\begin{aligned}\int_\Gamma \frac{1+i}{z(z-1)(z-i)}dz &= \int_\Gamma \frac{1-i}{z}dz + \int_\Gamma \frac{i}{z-1}dz - \int_\Gamma \frac{1}{z-i}dz \\ &= 2\pi i(1-i+i-1) = 0\end{aligned}$$

- If instead Γ has 0 and i in its interior and 1 in its exterior then

$$\int_\Gamma \frac{1+i}{z(z-1)(z-i)}dz = 2\pi i(1-i+0-1) = 2\pi$$

6 Cauchy's Integral Formula

Cauchy's Integral Formula

- Theorem 14. Let Γ be a simple closed positively oriented contour contained in a simply connected domain D. Let f be analytic in D. Then for any z_0 inside Γ

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$$

- Proof. We may replace Γ by the circle C_r given by $|z-z_0| = r$ and inside Γ.
- By the Deformation Invariance Theorem and the local version of Cauchy's integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$$

6.1 Cauchy's Integral Formula: Examples

Cauchy's Integral Formula: Examples

- Example. If Γ is the positively oriented circle $|z-3| = 5$ then

$$\int_\Gamma \frac{e^{2z}}{z(z-5i)}dz = \int_\Gamma \frac{\left(\frac{e^{2z}}{z-5i}\right)}{z}dz = 2\pi i\frac{e^0}{-5i} = -\frac{2\pi}{5}$$

- Example. Wth C_p being the circle $|z-p\pi| = \frac{\pi}{2}$, $p \in \mathbb{Z}$, we have $\int_{C_p} \frac{\cos z}{z-p\pi}dz = 2\pi i \cos p\pi = 2\pi i(-1)^p$.
- Thus $\int_{C_0} \frac{\cos z}{z}dz + \int_{C_1} \frac{\cos z}{z-\pi}dz = 0$. Also $\int_{C_0} \frac{\cos z}{z-\pi}dz = \int_{C_1} \frac{\cos z}{z}dz = 0$.
- Thus $\int_{C_0} \left(\frac{\cos z}{z} + \frac{\cos z}{z-\pi}\right)dz + \int_{C_1} \left(\frac{\cos z}{z} + \frac{\cos z}{z-\pi}\right)dz = 0$.
- If C is the circle $\left|z-\frac{\pi}{2}\right| = \pi$ then by deformation: $\int_c \left(\frac{\cos z}{z} + \frac{\cos z}{z-\pi}\right)dz = 0$.