Week 9: §§4.5-4.6

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1 Cauchy's Integral Formula and Its Consequences

1.1 Cauchy's Integral Formula and Theorem 15

Cauchy's Integral Formula and Theorem 15

- *Cauchy's integral formula*. Let Γ be simple, closed, positively oriented, and contained in a simply connected domain D. Let f be analytic in D. Then

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta \text{ for any } z \text{ inside } \Gamma$$

- Let g be continuous on the contour Γ. Then $G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta$ $(z \notin \Gamma)$, is analytic and $G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta-z)^2}d\zeta$.

- In fact $G^{(k)}$ is analytic for all $k \geq 0$ and $G^{(k)}(z) = k!\int_\Gamma \frac{g(\zeta)}{(\zeta-z)^{k+1}}d\zeta$ for each z not on Γ.

1.2 Theorems 16 and 17

Theorems 16 and 17

- Theorem 16. Let f be analytic in the domain D. Then derivatives of all orders exist and are analytic in D.

- Theorem 17. If $f = u + iv$ is analytic in a domain D, then $u, v \in C^\infty(D)$.

1.3 Morera's Theorem and The Generalized Cauchy Integral Formula

Morera's Theorem and The Generalized Cauchy Integral Formula

- Theorem 18. *Morera's Theorem*. If f is continuous in the domain D and if all loop integrals of f in D vanish, then f is analytic in D.

- Proof. According to the Equivalence Theorem f has an antiderivative F in D. As the derivative of F the function f is analytic.

- Theorem 19 (*Generalized Cauchy Integral Formula*). Let Γ be a simple closed positively oriented contour contained in a simply connected domain D. Let f be analytic in D. Then for any $n \in N$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \text{ for any } z \text{ inside } \Gamma$$

- Written differently

$$\int_\Gamma \frac{f(z)}{(z - z_0)^m} dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0)$$

1.4 The Generalized Cauchy Integral Formula: Examples

The Generalized Cauchy Integral Formula: Examples

- Example A. Let Γ be the circle $|z - 2| = 1$ (counterclockwise). Let Log be the principal branch of the logarithm. Then

$$\int_\Gamma \frac{\operatorname{Log}(z)}{(z-2)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \operatorname{Log}(z) \Big|_{z=2} = -\frac{\pi i}{4}$$

- Example B. Let now Γ be the circle $|z| = 1$ (counterclockwise). Then

$$\int_\Gamma \frac{\operatorname{Log}(z)}{z^3} dz = \int_{-\pi}^{\pi} \frac{\operatorname{Log}(e^{it})}{e^{3it}} ie^{it} dt = i \int_{-\pi}^{\pi} ite^{-2it} dt$$

- Integration by parts then gives us the result $\int_\Gamma \frac{\operatorname{Log}(z)}{z^3} dz = -\pi i$.
- The method of Example A could not be used. We had to resort to explicit calculation.

1.5 Cauchy Estimates

Cauchy Estimates

- Theorem 20. Let f be analytic inside and on the circle C_R given by $|z - z_0| = R$. Suppose $|f(z)| \le M$ for all z on C_R. Then for all $n \in N$

$$\left| f^{(n)}(z_0) \right| \le \frac{n! M}{R^n}$$

- Proof. By the generalized Cauchy formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

- Thus we get

$$\left| f^{(n)}(z_0) \right| = \frac{n!}{2\pi} \left| \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \le \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n! M}{R^n}$$

1.6 Liouville's Theorem and The Fundamental Theorem of Algebra

Liouville's Theorem and The Fundamental Theorem of Algebra

- Theorem 21. *Liouville's Theorem*. An entire and bounded function is constant.
- Proof. Let f be entire and satisfy $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By taking $n = 1$ in the Cauchy estimate

  $$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}$$

  we have for any z_0 and any R

  $$|f'(z_0)| \leq \frac{M}{R}$$

- But then $f'(z_0) = 0$ for any z_0, so that f is constant.
- Theorem 22. *The Fundamental Theorem of Algebra*. Every nonconstant polynomial has at least one zero.

1.7 The Fundamental Theorem of Algebra: Proof

The Fundamental Theorem of Algebra: Proof

- Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$. Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$.
- Then $f(z) = \frac{1}{P(z)}$ is entire. However, it is also bounded:
- We have as $|z| \to \infty$

  $$\frac{P(z)}{z^n} = a_n + a_{n-1} z^{-1} + \cdots + a_1 z^{-n+1} + a_0 z^{-n} \to a_n$$

- Thus there is an $R > 0$ such that for $|z| \geq R$

  $$\left|\frac{P(z)}{z^n}\right| \geq \frac{1}{2}|a_n|$$

- Thus for $|z| \geq R$

  $$|f(z)| = \frac{1}{|P(z)|} \leq \frac{2}{|z|^n |a_n|} \leq \frac{2}{R^n |a_n|}$$

- As a continuous function $|f|$ is also bounded on the closed disk $|z| \leq R$.
- By Liouville's theorem f is constant. Thus $n = 0$.

1.8 The Mean-value Property

The Mean-value Property

- Let f be analytic inside and on the circle C_R given by $|z - z_0| = R$. By Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

- C_R can be parametrized by $\zeta = z_0 + R\, e^{it}, t \in [0, 2\pi]$, so

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + R\, e^{it}\right)}{R\, e^{it}} R\, ie^{it} dt$$

- Thus f has the mean value-property:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\, e^{it}\right) dt$$

- Example C. Take $f = \exp$, $z_0 = 0$. We get the result

$$1 = e^0 = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(R\, e^{it}\right) dt$$

  which is not entirely obvious, but is confirmed by Maple.

1.9 The Mean-value Property and the Maximum Modulus Principle

The Mean-value Property and the Maximum Modulus Principle

- Lemma 1. Let f be analytic in the disk B given by $|z - z_0| \leq R$. If $\max_{z \in B} |f(z)| = |f(z_0)|$ then $|f|$ is constant in B.

- Proof. Since for any $r \leq R$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + r\, e^{it}\right) \right| dt$$

  and since $\left| f\left(z_0 + r\, e^{it}\right) \right| \leq |f(z_0)|$ for all t, we in fact must have $\left| f\left(z_0 + r\, e^{it}\right) \right| = |f(z_0)|$ for all t and all $r \leq R$.

- Theorem 23. *The Maximum Modulus Principle*. Let f be analytic in the domain D. Suppose $|f(z)|$ attains its maximum at a point $z_0 \in D$. Then f is constant.

- Remark. Notice that the conclusion is that f is constant, not only $|f|$.

1.10 The Maximum Modulus Principle I

The Maximum Modulus Principle I

- Proof. Let $z_1 \in D$ be any other point than the point at which $|f(z)|$ attains its maximum z_0.
- Connect z_0 and z_1 by a contour γ lying in D and parametrized by $z = \phi(t), t \in [a, b]$ with $\phi(a) = z_0$ and $\phi(b) = z_1$.
- The real-valued function $|f(\phi(t))|, t \in [a, b]$ is constant on some interval $[a, c]$ $(c > a)$ by Lemma 1.
- Let $c_m = \max\{c \in [a, b] \, | \, |f(\phi(t))| = |f(z_0)| \text{ for } t \in [a, c]\}$.
- We shall show that $c_m = b$. Let B be a disk centered at $w = \phi(c_m)$ and contained in D.
- $|f|$ also attains its maximum at w. Thus by Lemma1 $|f|$ is constant in B. So necessarily $c_m = b$, and $|f(z_1)| = |f(z_0)|$.
- But if $|f|$ is constant in D, so is f (see the lecture notes for week 3).

1.11 The Maximum Modulus Principle II

The Maximum Modulus Principle II

- Theorem 24. *The Maximum Modulus Principle: Version 2*. Let f be analytic in the bounded domain D. Suppose f is continuous on the closure $\overline{D}$. Then f attains its maximum modulus on the boundary of D.
- Proof. As a continuous function $|f|$ attains its maximum at some point z_0 of the compact set $\overline{D}$. If $z_0 \in D$ then f is constant in D and by continuity also in $\overline{D}$.
- Theorem 26. If ϕ is harmonic in a simply connected domain D and attains its maximum or minimum at some point in D, then ϕ is constant.
- Proof. ϕ is the real part of an analytic function f. Suppose $z_0 \in D$ is a maximum point for ϕ. Then $\left|e^{f(z)}\right| = e^{\phi(z)}$ attains its maximum at z_0. However, $\exp \circ f$ is analytic, so must be constant in D. Therefore ϕ is constant.
- A minimum is handled by considering $-\phi$ instead.