Week 11: §§5.5–5.7

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1 Laurent Series, Zeros, and Singularities

1.1 Laurent Series

Laurent Series

- Theorem 14. Let f be analytic in the annulus $r < |z - z_0| < R$. Then f can be expressed as the sum of two series

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$

  where both series are convergent in the annulus.

- The convergence is uniform in any compact subset of the annulus.

- The coefficients a_j are given by

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta$$

  where C is any positively oriented simple closed contour lying in the annulus and having z_0 in its interior.

1.2 Laurent Series. Proof of Theorem 14, I

Laurent Series. Proof of Theorem 14, I

- Without loss of generality we may assume that $z_0 = 0$.

- Let $r < R_1 < \rho_1 < \rho_2 < R_2 < R$. Let z satisfy $\rho_1 \le |z| \le \rho_2$.

- Let C_1 and C_2 be the circles with center 0 and radii R_1 and R_2 both positively oriented.

- By Cauchy's formula (See Maple animation in Lweek11.mw):

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

- When $|z| \leq \rho_2$ and $|\zeta| = R_2$ the series $\sum_{j=0}^{\infty} \frac{z^j}{\zeta^{j+1}}$ is uniformly convergent as a function of ζ with sum $\frac{1}{\zeta} \cdot \frac{1}{1-\frac{z}{\zeta}} = \frac{1}{\zeta - z}$.
- We may integrate term by term:

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{C_2} f(\zeta) \frac{z^j}{\zeta^{j+1}} d\zeta$$

1.3 Laurent Series. Proof of Theorem 14, II

Laurent Series. Proof of Theorem 14, II

- Rearranging slightly we get

$$\sum_{j=0}^{\infty} \frac{z^j}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta^{j+1}} d\zeta = \sum_{j=0}^{\infty} a_j z^j$$

- The resulting series is uniformly convergent as a function of z.
- When $|z| \geq \rho_1$ and $|\zeta| = R_1$ the series $\sum_{j=0}^{\infty} \frac{\zeta^j}{z^{j+1}}$ is uniformly convergent as a function of ζ with sum $\frac{1}{z} \cdot \frac{1}{1-\frac{\zeta}{z}} = -\frac{1}{\zeta - z}$.
- We may integrate term by term:

$$-\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{C_1} f(\zeta) \frac{\zeta^j}{z^{j+1}} d\zeta$$

1.4 Laurent Series. Proof of Theorem 14, III

Laurent Series. Proof of Theorem 14, III

- Rearranging slightly we get

$$\begin{aligned} & \sum_{j=0}^{\infty} \frac{z^{-j-1}}{2\pi i} \int_{C_1} f(\zeta) \zeta^j d\zeta \\ = & \sum_{j=1}^{\infty} \frac{z^{-j}}{2\pi i} \int_{C_1} f(\zeta) \zeta^{j-1} d\zeta \\ = & \sum_{j=1}^{\infty} a_{-j} z^{-j} \end{aligned}$$

- The resulting series is uniformly convergent as a function of z.
- By deformation invariance we may replace the circles C_1 and C_2 in the formulas for a_j by any simple positively oriented closed contour C lying in the annulus and having z_0 in its interior.

1.5 Uniqueness of Laurent Series. Theorem 15

Uniqueness of Laurent Series. Theorem 15

- Let $\sum_{j=0}^{\infty} c_j (z-z_0)^j$ be convergent for $|z-z_0| < R$ and let $\sum_{j=1}^{\infty} c_{-j} (z-z_0)^{-j}$ be convergent for $|z-z_0| > r$.
- If $r < R$ let $f(z)$ be the sum of the the two series for $r < |z-z_0| < R$. Then f is analytic in that annulus.
- The Laurent expansion for f is given by $\sum_{j=-\infty}^{\infty} c_j (z-z_0)^j$.
- Proof. Let C be any simple closed positively oriented contour lying in the annulus and having z_0 in its interior. We show that $c_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-z_0)^{j+1}} d\zeta$.
- We replace $f(\zeta)$ by its definition. By uniform convergence on C we may interchange integration and summation.
- The result then follows from $\frac{1}{2\pi i} \oint_C \frac{d\zeta}{(\zeta-z_0)^n} = \begin{cases} 1 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$.

1.6 Remark on Odd and Even Functions

Remark on Odd and Even Functions

- Let f be analytic in the annulus $r < |z| < R$ and have Laurent coefficients c_j.
- If f is even then $c_{2k-1} = 0$ for all k. If f is odd then $c_{2k} = 0$ for all k.
- Proof. Let C be the circle given by $z = \rho e^{it}, t \in [0, 2\pi]$, where $r < \rho < R$. Suppose f is even. Then $2\pi i c_{2k-1} = \oint_C \frac{f(\zeta)}{\zeta^{2k}} d\zeta = i \int_0^{2\pi} \frac{f(\rho e^{it})}{\rho^{2k-1} e^{it(2k-1)}} dt$.
- But $f(\rho e^{it}) = f(-\rho e^{it}) = f\left(\rho e^{i(t+\pi)}\right)$. Making the change of variable $\tau = t + \pi$ we therefore get

$$\begin{aligned} 2\pi i c_{2k-1} &= i \int_{\pi}^{3\pi} \frac{f(\rho e^{i\tau})}{\rho^{2k-1} e^{i(\tau-\pi)(2k-1)}} d\tau \\ &= -i \int_{\pi}^{3\pi} \frac{f(\rho e^{i\tau})}{\rho^{2k-1} e^{i\tau(2k-1)}} d\tau = -2\pi i c_{2k-1} \end{aligned}$$

- Thus $c_{2k-1} = 0$. The proof is quite similar if f is odd instead.

1.7 Example 1

Example 1

- The Laurent series in the region $|z-2i| > 1$ for $f(z) = \frac{z^3+8i}{z-i}$.

- We let $w = (z-2i)^{-1}$, so $z = 2i + \frac{1}{w}$. Then

$$\frac{1}{z-i} = \frac{1}{i+\frac{1}{w}} = \frac{w}{1+iw} = \sum_{j=0}^{\infty} w\,(-iw)^j = \sum_{j=0}^{\infty} (-i)^j\, w^{j+1}$$

- Since $z^3 + 8i = w^{-3} + 6iw^{-2} - 12w^{-1}$ we find

$$\begin{aligned}
f(z) &= \left(w^{-3} + 6iw^{-2} - 12w^{-1}\right) \sum_{j=0}^{\infty} (-i)^j\, w^{j+1} \\
&= \sum_{j=0}^{\infty} (-i)^j\, w^{j-2} + 6i \sum_{j=0}^{\infty} (-i)^j\, w^{j-1} - 12 \sum_{j=0}^{\infty} (-i)^j\, w^j \\
&= w^{-2} + 5iw^{-1} + \sum_{j=0}^{\infty} (-1+6-12)\,(-i)^j\, w^j \\
&= (z-2i)^2 + 5i\,(z-2i) - 7 \sum_{j=0}^{\infty} (-i)^j\, (z-2i)^{-j}
\end{aligned}$$

1.8 Example 2 a (from the book)

Example 2 a (from the book)

- The Laurent series in the region $|z| < 1$ for

$$\frac{1}{(z-1)(z-2)}$$

- The function is analytic in the region so has a Maclaurin expansion:

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j + \sum_{j=0}^{\infty} z^j \\
&= \sum_{j=0}^{\infty} \left(1 - 2^{-j-1}\right) z^j
\end{aligned}$$

- The Laurent series expansion is just a Maclaurin series expansion.

1.9 Example 2 b

Example 2 b

- We still consider

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

- but in the region $1 < |z| < 2$. We find

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{z}} = \sum_{j=0}^{\infty} z^{-j-1}$$

- With the result from (a) above we find

$$\frac{1}{(z-1)(z-2)} = -\sum_{j=0}^{\infty} 2^{-j-1} z^j - \sum_{j=0}^{\infty} z^{-j-1}$$

1.10 Example 2 c

Example 2 c

- We still consider

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

- now finally in the region $2 < |z|$. We find as above

$$\frac{1}{z-1} = \sum_{j=0}^{\infty} z^{-j-1}$$

- and now also

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{j=0}^{\infty} 2^j z^{-j-1}$$

- Combining these results we get

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} 2^j z^{-j-1} - \sum_{j=0}^{\infty} z^{-j-1} = \sum_{j=0}^{\infty} \left(2^j - 1\right) z^{-j-1}$$

1.11 Example 3

Example 3

- Let f be analytic at 0 and have the Maclaurin expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

valid for $|z| < \rho$.

- Then the function $g(z) = f\left(\frac{1}{z}\right)$ will in the region $|z| > \frac{1}{\rho}$ have the Laurent expansion

$$g(z) = \sum_{j=0}^{\infty} a_j z^{-j}$$

- $f(z) = \sin(z) + \sin\left(\frac{1}{z}\right)$ is analytic for $z \neq 0$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n-1}}{(2n+1)!}$$

1.12 Zeros and Singularities I

Zeros and Singularities I

- Definition 7. z_0 is called a *zero of order m* for f if f is analytic at z_0 and $f^{(k)}(z_0) = 0$ for $k = 0, 1, \ldots, m-1$, but $f^{(m)}(z_0) \neq 0$.
- If z_0 is a zero of order m then the Taylor series for f around z_0 can be written $f(z) = (z - z_0)^m \sum_{j=0}^{\infty} a_{m+j}(z - z_0)^j$ where $a_m = \frac{1}{m!} f^{(m)}(z_0) \neq 0$.
- Theorem 16. Let f be analytic at z_0. The f has a zero of order m at z_0 iff f can be written as $f(z) = (z - z_0)^m g(z)$ where g is analytic at z_0 and $g(z_0) \neq 0$.
- Zeros of order 1 are called *simple* zeros.
- Corollary 3. If z_0 is a zero for f then either f is identically zero in a disk about z_0 or different from zero in a punctured disk about z_0.
- Thus if a nonconstant analytic function has a zero it is isolated.

1.13 Zeros and Singularities II

Zeros and Singularities II

- Definition 8. Let z_0 be an isolated singularity for the analytic function f. Let the Laurent expansion for f in $0 < |z - z_0| < R$ be $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$. Then
- If $a_j = 0$ for all $j < 0$ then z_0 is called a *removable singularity* of f.
- If $a_{-m} \neq 0$ for some $m > 0$ but $a_j = 0$ for all $j < -m$ then z_0 is called a *pole of order m* for f.
- If $a_j \neq 0$ for infinitely many negative values of j then z_0 is called an *essential singularity* of f.
- Lemma 5. If f has a removable singularity at z_0 then
  - f is bounded in some punctured neighborhood of z_0
  - the limit $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$.
  - f can be redefined at z_0 so that it becomes analytic at z_0.

1.14 Zeros and Singularities III

Zeros and Singularities III

- Lemma 6. If f has a pole of order m at z_0, then $\left|(z - z_0)^k f(z)\right| \to \infty$ as $z \to z_0$ for $k < m$ whereas $(z - z_0)^m f(z)$ has a removable singularity at z_0.

- Lemma 7. A function f has a pole of order m at z_0 iff in some punctured disk about z_0 there is an analytic function g with $g(z_0) \neq 0$ such that $f(z) = \frac{g(z)}{(z-z_0)^m}$.

- Lemma 8. If f has a zero of order m at z_0, then $\frac{1}{f}$ has a pole of order m at z_0. If f has a pole of order m at z_0 then $\frac{1}{f}$ has a removable singularity at z_0 and by defining $\left(\frac{1}{f}\right)(z_0) = 0$ the function $\frac{1}{f}$ has a zero of order m at z_0.

- Theorem 17. *Picard's Theorem*. If f has an essential singularity at z_0, then f takes any complex value possibly with one exception in any neighborhood of z_0.

1.15 Zeros and Singularities IV

Zeros and Singularities IV

- Theorem 18. If f has an isolated singularity at z_0, then we have:

- z_0 is a removable singularity $\Leftrightarrow$ $|f|$ is bounded near z_0 $\Leftrightarrow$ $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$ $\Leftrightarrow$ f can be redefined at z_0 so that f is analytic at z_0.

- z_0 is a pole $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to z_0$ $\Leftrightarrow$ f can be written $f(z) = g(z) / (z-z_0)^m$ for some $m > 0$ and some analytic function g with $g(z_0) \neq 0$.

- z_0 is an essential singularity $\Leftrightarrow$ $|f|$ is not bounded near z_0 but $|f(z)|$ does not tend to infinity as $z \to z_0$ $\Leftrightarrow$ f assumes every complex value with possibly one exception in every neighborhood of z_0.

1.16 The Point at Infinity I

The Point at Infinity I

- Definition. Let f be analytic in the exterior of some circle given by $|z| > R$. f is said to be analytic at ∞ if the function $w \mapsto f\left(\frac{1}{w}\right)$ which is analytic for $0 < |w| < \frac{1}{R}$ is also analytic at $w = 0$ or has a removable singularity at $w = 0$.

- Notice that f is still considered a function with values in $\mathbb{C}$, i.e. ∞ is not a value taken on by f.

- f has a pole of order m at ∞ if $w \mapsto f\left(\frac{1}{w}\right)$ has a pole of order m at $w = 0$.

- f has an essential singularity at ∞ if $w \mapsto f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$.

- Theorem. If f has an isolated singularity at ∞, then we have:

- f is analytic at ∞ $\Leftrightarrow$ $|f|$ is bounded near ∞ $\Leftrightarrow$ $\lim_{z \to \infty} f(z)$ exists in $\mathbb{C}$.

- f has a pole at $\infty \Leftrightarrow |f(z)| \to \infty$ as $z \to \infty$.
- f has an essential singularity at $\infty \Leftrightarrow |f|$ is not bounded near ∞ but $|f(z)|$ does not tend to infinity as $z \to \infty$.

1.17 The Point at Infinity II Examples

The Point at Infinity II Examples

- $\exp z, \cos z$ and $\sin z$ have essential singularities at ∞.
- A polynomial of degree $n > 0$ has a pole of order n at ∞.
- The Möbius transformation $\frac{az+b}{cz+d}$ is analytic at ∞ if $c \neq 0$.
- If an entire function is analytic at ∞ then it is constant.
- If an entire function has a pole of order m at ∞ then it is a polynomial of degree m.