Week 12: §6.1 and §6.3

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November 27, 2008

1 The Residue Theorem and Improper Integrals

1.1 The Residue

The Residue

- Let f have an isolated singularity at z_0. The coefficient a_{-1} of $(z-z_0)^{-1}$ in the Laurent expansion for f in an annulus $0 < |z-z_0| < R$ is the *residue of f at z_0* and is denoted by $\operatorname{Res}(f;z_0)$ or $\operatorname{Res}(z_0)$.
- If Γ is a simple closed positively oriented contour having z_0 in its interior and lying in the annulus then $\int_\Gamma f(z)\,dz = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(z_0)$.
- Example A. Compute the integral $\oint_{|z|=1/2} \frac{-\operatorname{Log}(1-z)}{z^4} dz$ (counterclockwise).
- The residue of the function $f(z) = \frac{-\operatorname{Log}(1-z)}{z^4}$ at 0 is found from

$$\frac{-\operatorname{Log}(1-z)}{z^4} = z^{-4}\sum_{n=1}^{\infty}\frac{z^n}{n} = \sum_{n=1}^{\infty}\frac{z^{n-4}}{n} = \sum_{n=-3}^{\infty}\frac{z^n}{n+4}$$

- We see that $\operatorname{Res}(0) = \frac{1}{3}$ so that $\oint_{|z|=1/2}\frac{-\operatorname{Log}(1-z)}{z^4}dz = 2\pi i\frac{1}{3} = \frac{2}{3}\pi i$.

1.2 Computation of the Residue I

Computation of the Residue I

- If f has a simple pole at z_0 so that $f(z) = \sum_{j=-1}^{\infty} a_j (z-z_0)^j$ then

$$\operatorname{Res}(z_0) = a_{-1} = \lim_{z\to z_0}(z-z_0)f(z)$$

- Example 2 (in the book). Let f be the ratio of two functions h and g both analytic at z_0, $f(z) = \frac{h(z)}{g(z)}$.
- If g has a simple zero at z_0 and $h(z_0) \neq 0$ then

$$\operatorname{Res}(z_0) = a_{-1} = \lim_{z\to z_0}(z-z_0)\frac{h(z)}{g(z)} = \frac{h(z_0)}{g'(z_0)}$$

- Example B. The singularities of $\tan z = \frac{\sin z}{\cos z}$ are $z = \frac{\pi}{2} + p\pi, p \in \mathbb{Z}$. They are all simple and $\sin\left(\frac{\pi}{2} + p\pi\right) = (-1)^p \neq 0$.
- Since $\frac{d}{dz}\cos z = -\sin z$ we get for all $p \in \mathbb{Z}$

$$\text{Res}\left(\frac{\pi}{2} + p\pi\right) = -\frac{\sin\left(\frac{\pi}{2} + p\pi\right)}{\sin\left(\frac{\pi}{2} + p\pi\right)} = -1$$

1.3 Computation of the Residue II

Computation of the Residue II

- Theorem 1. If f has a pole of order m at z_0, then

$$\text{Res}\,(f; z_0) = \frac{1}{(m-1)!}\lim_{z \to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]$$

- Example C. We find the residues of all singularities in the disk $|z| < 4$ of

$$f(z) = \frac{\text{Log}\,(4 - z)}{z^2 (z-3)^4}$$

- f has a pole of order 2 at 0 and pole of order 3 at 3 (not order 4, since $\text{Log}(4 - z)$ has a simple zero at 3) at 3.
- $\text{Res}(0) = \lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\frac{d}{dz}\left[\frac{\text{Log}(4-z)}{(z-3)^4}\right] = \lim_{z\to 0}\left[\frac{1}{(z-3)^4(z-4)} - 4\frac{\text{Log}(4-z)}{(z-3)^5}\right] = \frac{8}{243}\ln 2 - \frac{1}{324}$.
- $\text{Res}(3) = \frac{1}{2}\lim_{z\to 3}\frac{d^2}{dz^2}\left[(z-3)^3 f(z)\right] = \frac{1}{2}\lim_{z\to 3}\frac{d^2}{dz^2}\left[\frac{\text{Log}(4-z)}{z^2(z-4)}\right] = -\frac{1}{27}$ (used Maple for the last step).

1.4 Cauchy's Residue Theorem

Cauchy's Residue Theorem

- Theorem 2. Let Γ be a simple closed positively oriented contour and suppose f is analytic inside and on Γ except at the points $z_1, z_2, \ldots, z_n$ inside Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\,(f; z_k)$$

- Proof. We may write

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n}\int_{C_k} f(z)\,dz$$

where C_k is a positively oriented circle with center at z_k and a radius r_k chosen small enough to ensure that the disks $|z - z_k| \leq r_k$ don't intersect. See Maple animation.

- In the disk $|z - z_k| \leq r_k$ we write $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_k)^j$.
- Then $\int_{C_k} f(z)\,dz = 2\pi i a_{-1} = 2\pi i \text{Res}(f; z_k)$.

1.5 Cauchy's Residue Theorem, Example D

Cauchy's Residue Theorem, Example D

- Let C be the circle $|z-1-i| = \frac{5}{2}$ traversed couterclockwise. We evaluate the integral
$$\oint_C \frac{z^2+z+1}{(z-1)(z-2i)(z+2-i)}dz$$
- The integrand $f(z)$ is analytic inside C except at the two simple poles 1 and $2i$.
- Thus $\oint_C f(z)\,dz = 2\pi i\,(\text{Res}(1) + \text{Res}(2i))$.
- $\text{Res}(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z^2+z+1}{(z-2i)(z+2-i)} = \frac{3}{(1-2i)(3-i)} = \frac{3}{50} + \frac{21}{50}i$.
- $\text{Res}(2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{z^2+z+1}{(z-1)(z+2-i)} = \frac{-3+2i}{(-1+2i)(2+i)} = \frac{18}{25} + \frac{1}{25}i$.
- Thus $\oint_C f(z)\,dz = 2\pi i\left(\frac{3}{50} + \frac{21}{50}i + \frac{18}{25} + \frac{1}{25}i\right) = \left(-\frac{23}{25} + \frac{39}{25}i\right)\pi$.

1.6 Cauchy's Residue Theorem, Example E

Cauchy's Residue Theorem, Example E

- We evaluate the integral
$$\oint_{|z|=2}\left(\tan z + \frac{\text{Log}(4-z)}{z^2(z-3)^4}\right)dz$$
- First we write the integral as the sum
$$\oint_{|z|=2}\tan z dz + \oint_{|z|=2}\frac{\text{Log}(4-z)}{z^2(z-3)^4}dz$$
- Then we use residues to find
$$2\pi i\left(\text{Res}\left(\tan z; -\frac{\pi}{2}\right) + \text{Res}\left(\tan z; \frac{\pi}{2}\right) + \text{Res}\left(\frac{\text{Log}(4-z)}{z^2(z-3)^4}; 0\right)\right)$$
- But these residues we found earlier, so the given integral has the value
$$2\pi i\left(-1 + (-1) + \frac{8}{243}\ln 2 - \frac{1}{324}\right) = 2\pi i\left(\frac{8}{243}\ln 2 - \frac{649}{324}\right)$$

1.7 Integrals over $\mathbb{R}$ by Residues I

Integrals over $\mathbb{R}$ by Residues I

- Suppose the limit
$$\lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx$$
exists, then its value is called the *Cauchy principal value* of the integral of f over $\mathbb{R}$ and is denoted by
$$p.v.\int_{-\infty}^{\infty} f(x)\,dx$$

- If both of the integrals
$$\int_{0}^{\infty} f(x)\,dx \text{ and } \int_{-\infty}^{0} f(x)\,dx$$
exist, then the Cauchy principle value does too and
$$p.v.\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx$$

- The converse is false. See next slide.

1.8 Integrals over $\mathbb{R}$ by Residues II

Integrals over $\mathbb{R}$ by Residues II

- Example. We have $p.v.\int_{-\infty}^{\infty}\sin x\;dx = 0$, but neither $\int_{0}^{\infty}\sin x\;dx$ nor $\int_{-\infty}^{0}\sin x\;dx$ exists.

- Example. $p.v.\int_{-\infty}^{\infty}\cos x\,dx$ does not exist: $\int_{-\rho}^{\rho}\cos x\,dx = 2\sin\rho$. No limit as $\rho\to\infty$.

- Example F. We find $\int_{-\infty}^{\infty}\frac{x^4dx}{x^6+64}$, which clearly is convergent in the usual sense.

- Consider $f(z) = \frac{z^4}{z^6+64}$ in $\mathbb{C}$. It has the 6 simple poles $\pm\sqrt{3}\pm i$ (all 4 combinations) and $\pm 2i$.

- Let Γ_ρ consist of the real line segment $[-\rho,\rho]$ and the circular arc C_ρ^+ parametrized by $z=\rho e^{it}, t\in[0,\pi]$.

- For $\rho > 2$ the poles in the upper half-plane are inside Γ_ρ. Thus
$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i\left(\operatorname{Res}(2i) + \operatorname{Res}\left(\sqrt{3}+i\right) + \operatorname{Res}\left(-\sqrt{3}+i\right)\right)$$

1.9 Integrals over $\mathbb{R}$ by Residues III

Integrals over $\mathbb{R}$ by Residues III

- The residues are easily calculated, so for $\rho > 2$

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i\left(-\frac{i}{12} + \frac{\sqrt{3}-i}{24} + \frac{-\sqrt{3}-i}{24}\right) = \frac{1}{3}\pi$$

- Now we show that the contribution from the circular arc tends to zero as $\rho \to \infty$.
- We have

$$\begin{aligned}\left|\int_{C_\rho^+} f(z)\,dz\right| &= \left|\int_0^\pi \frac{\rho^4 e^{4it} i\rho e^{it}}{\rho^6 e^{6it} + 64} dt\right| \\ &\leq \int_0^\pi \frac{\rho^5}{\rho^6 - 64} dt = \pi \frac{\rho^5}{\rho^6 - 64} \\ &= \pi \frac{1}{\rho - \frac{64}{\rho^5}} \to 0 \text{ as } \rho \to \infty\end{aligned}$$

- We conclude that $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6+64} = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{x^4 dx}{x^6+64} = \frac{\pi}{3}$.

1.10 Integrals over $\mathbb{R}$ by Residues IV

Integrals over $\mathbb{R}$ by Residues IV

- Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

$$\lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\,dz = 0$$

  where C_ρ^+ is defined as in Example F.

- Example G. Consider $\int_{-\infty}^{\infty} \frac{3x^5+x^4}{x^6+64} dx$. The improper integral is not convergent in the usual sense, but the Cauchy principal value exists:
- We have

$$\begin{aligned} p.v. \int_{-\infty}^{\infty} \frac{3x^5+x^4}{x^6+64} dx &= \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{3x^5+x^4}{x^6+64} dx \\ &= \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{3x^5}{x^6+64} dx + \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{x^4}{x^6+64} dx \end{aligned}$$

- $\int_{-\infty}^{\infty} \frac{3x^5}{x^6+64} dx = 0$ since $\frac{3x^5}{x^6+64}$ is an odd function of x.
- Thus $p.v. \int_{-\infty}^{\infty} \frac{3x^5+x^4}{x^6+64} dx = \frac{\pi}{3}$.

1.11 Example 3 from §6.3, I

Example 3 from §6.3, I

- We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.
- Since $0 \le \frac{e^{ax}}{1+e^x} \le e^{(a-1)x}$ the improper integral $\int_0^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent.
- Since $0 \le \frac{e^{ax}}{1+e^x} \le e^{ax}$ the improper integral $\int_{-\infty}^{0} \frac{e^{ax}}{1+e^x} dx$ is convergent.
- Thus certainly $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent when $0 < a < 1$.
- Let Γ_ρ be the contour consisting of γ_1: $z = x, x \in [-\rho, \rho]$, γ_2: $z = \rho + iy, y \in [0, 2\pi]$, γ_3: $z = x + 2\pi i, x \in [-\rho, \rho]$ (but opposite), γ_4: $z = -\rho + iy, y \in [0, 2\pi]$ (again opposite).
- $f(z) = \frac{e^{az}}{1+e^z}$ has the simple pole πi inside the contour.
- Thus $\int_{\Gamma_\rho} f(z)\, dz = 2\pi i \text{Res}(\pi i) = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i}$.
- We have $\left| \int_{\gamma_2} f(z)\, dz \right| = \left| \int_0^{2\pi} \frac{e^{a(\rho+iy)}}{1+e^{\rho+iy}} i dy \right| \le \int_0^{2\pi} \frac{e^{a\rho}}{e^\rho - 1} dy = 2\pi \frac{e^{a\rho}}{e^\rho - 1} \to 0$ as $\rho \to \infty$.

1.12 Example 3, II

Example 3, II

- We have $\left| \int_{\gamma_4} f(z)\, dz \right| = \left| \int_0^{2\pi} \frac{e^{a(-\rho+iy)}}{1+e^{-\rho+iy}} i dy \right| \le \int_0^{2\pi} \frac{e^{-a\rho}}{1-e^{-\rho}} dy = 2\pi \frac{e^{-a\rho}}{1-e^{-\rho}} \to 0$ as $\rho \to \infty$.
- Now

$$\begin{aligned} \int_{\gamma_3} f(z)\, dz &= -\int_{-\rho}^{\rho} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx = -e^{a2\pi i} \int_{-\rho}^{\rho} \frac{e^{ax}}{1+e^x} dx \\ &= -e^{a2\pi i} \int_{\gamma_1} f(z)\, dz \end{aligned}$$

- Thus

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{\rho \to \infty} \int_{\gamma_1} f(z)\, dz = \frac{-2\pi i e^{a\pi i}}{1 - e^{a2\pi i}} = \frac{\pi}{\sin(a\pi)}$$