Complex Numbers

Preben Alsholm

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- We have $N \subset Z \subset Q \subset R \subset C$.

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What is a Cauchy sequence?

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- C has the very important property that *every polynomial of degree* ≥ 1 *has at least one root*. (The Fundamental Theorem of Algebra).

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- It follows that $i^2 = -1$.

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- The usual method of calculating $\frac{b}{a}$:

$$\begin{aligned}\frac{2+3i}{-4+7i} &= \frac{(2+3i)(-4-7i)}{(-4+7i)(-4-7i)} = \frac{(2+3i)(-4-7i)}{(-4)^2-(7i)^2} \\ &= \frac{(2+3i)(-4-7i)}{16+49} = \frac{13-26i}{65} = \frac{1}{5} - \frac{2}{5}i\end{aligned}$$

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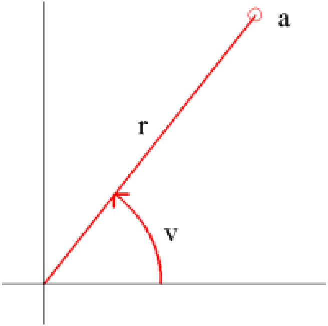
- The triangle inequality: $|a+b| \leq |a| + |b|$

Polar form I

- Let $r = |a|$ and v be an angle measured from the positive real axis to the line connecting 0 and a (measured positive in the counterclockwise direction).



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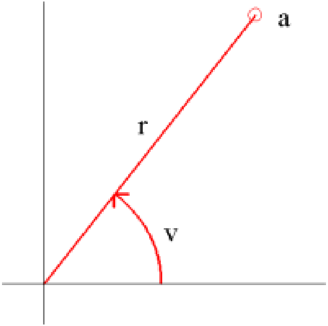

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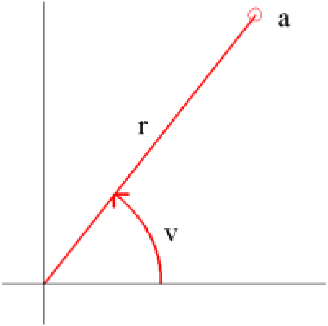

- v is an argument for a. Notation: $\arg(a)$. The set of arguments for a is $\{v + p2\pi \,|\, p \in Z\}$.
- Any complex number can be written in polar form: $a = r \cdot (\cos v + i \sin v)$, where r is the modulus and v is an argument of a.

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- $\arg\left(\frac{a}{b}\right) = \arg a - \arg b$
- These must be properly understood: Thus $\arg(ab) = \arg a + \arg b$ means that one of the arguments for ab is obtained by adding an argument for a and an argument for b.

The Complex Exponential

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- $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ for all $z_1, z_2 \in C$, i.e. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.

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- $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ for all $z_1, z_2 \in C$, i.e. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.
- Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\left|e^{z_1} \cdot e^{z_2}\right| &= \left|e^{z_1}\right| \cdot \left|e^{z_2}\right| = \left|e^{x_1+iy_1}\right| \cdot \left|e^{x_2+iy_2}\right| = e^{x_1} \cdot e^{x_2} \\
&= e^{x_1+x_2} = \left|e^{x_1+x_2+i(y_1+y_2)}\right| = \left|e^{z_1+z_2}\right| \\
\arg\left(e^{z_1} \cdot e^{z_2}\right) &= \arg\left(e^{z_1}\right) + \arg\left(e^{z_2}\right) = \arg\left(e^{x_1+iy_1}\right) + \arg\left(e^{x_2+iy_2}\right) \\
&= y_1 + y_2 = \arg\left(e^{x_1+x_2+i(y_1+y_2)}\right) = \arg\left(e^{z_1+z_2}\right)
\end{aligned}$$

The polar form once more

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$$a = r\left(\cos v + i \sin v\right)$$

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- Example. The polar form for $-\sqrt{3} - i$. Modulus: $\sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$. An argument is $-\frac{5\pi}{6}$. Thus

$$-\sqrt{3} - i = 2\exp\left(-i\frac{5\pi}{6}\right) = 2e^{-i\frac{5\pi}{6}}$$

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- Example.

$$\begin{aligned}\cos 3x &= \mathrm{Re}\,(\cos 3x + i\sin 3x) = \mathrm{Re}\left((\cos x + i\sin x)^3\right) \\ &= \mathrm{Re}\left(\cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x\right) \\ &= \cos^3 x - 3\cos x\sin^2 x = \cos^3 x - 3\cos x\left(1-\cos^2 x\right) \\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

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\end{aligned}$$

- By replacing Re with Im above we get the formula

$$\begin{aligned}
\sin 3x &= 3\cos^2 x\sin x - \sin^3 x \\
&= 3\left(1 - \sin^2 x\right)\sin x - \sin^3 x \\
&= -4\sin^3 x + 3\sin x
\end{aligned}$$