Week 2, 1.5-1.7 and 2.1-2.2

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- Proof: Let $z = \rho e^{i\theta}$, with $\rho \geq 0$ and $\theta \in \mathbb{R}$. By inserting this in (1) we get

$$\left(\rho e^{i\theta}\right)^n = re^{iv} \text{ and thus } \rho^n e^{in\theta} = re^{iv}$$

The two sides of this equation are polar forms of the same number. Therefore $\rho^n = r$ and $n\theta = v + p2\pi$, where $p \in \mathbb{Z}$.

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- ω_n^p (where $1 \leq p \leq n-1$) is a primitive root of unity iff $\gcd(p, n) = 1$, i.e. iff p and n are relatively prime.

The Quadratic Equation

- Let $a, b, c \in \mathbb{C}$, and assume $a \neq 0$. Then

$$az^2 + bz + c = a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right)$$

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- Thus the roots of $az^2 + bz + c = 0$ are given by

$$z = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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- S is *compact* if it is closed and bounded.
- S is a *region* if it is the union of a domain D and some, none, or all of the boundary points of D.

Theorem 1 (in 1.6). The Riemann Sphere.

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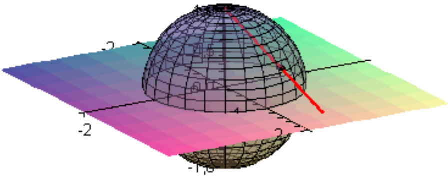

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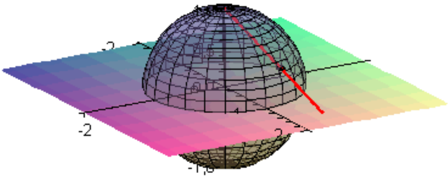

- In Maple's *conformal3d* the sphere is placed on top of the complex plane.

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- The stereographic projection of $z \in \mathbb{C}$ onto the unit sphere is given by (see Maple worksheet for Ch. 1)

$$(x_1, x_2, x_3) = \left(\frac{2\,\mathrm{Re}\, z}{|z|^2 + 1}, \frac{2\,\mathrm{Im}\, z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

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- The *chordal distance* between z and w is the ordinary distance between their stereographic projections. It is given by

$$\chi(z, w) = \frac{2\,|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \quad \chi(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}$$

Limits and Continuity I

- Let $A \subseteq \mathbb{C}$ and let $f : A \rightarrow \mathbb{C}$ be a mapping from A into $\mathbb{C}$. f can be written

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- Suppose z_0 is an accumulation point for the domain of definition A of f. We say that $\lim_{z\rightarrow z_0} f(z) = w_0$ if

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- Let f and g be continuous in A and $z_0 \in A$. If $g(z_0) = 0$ but $h(z) = \frac{f(z)}{g(z)}$ has a limit as $z \to z_0$, then h has a *removable discontinuity* at z_0.

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Limits and Continuity III

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- If we work in the extended complex plane we define continuity by the requirement $\lim_{z\to z_0} f(z) = f(z_0)$ even when $z_0 = \infty$ and even when $f(z_0) = \infty$.