Week 4, 3.2, 3.3, and 3.5

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September 25, 2008

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- The well-known identities known from the real trig. functions still hold.

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- The *principal value* of the logarithm is $Log\ z = \ln|z| + iArg\ z$.
- Theorem. *Log* is analytic on $D^* = \mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$ and $\frac{d}{dz} Log\ z = \frac{1}{z}$ for all $z \in D^*$.

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- We shall see later that an analytic function maps an open set onto an open set: The *open mapping property* of analytic functions (Theorem 5 p. 363). It does not require f to be 1-1.

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- Example. Determine a branch of $f(z) = \log\left(z^3 - 2\right)$ that is analytic at 0.

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- Notice that the values of $z^{\frac{m}{n}}$ are the solutions ζ to the binomial equation $\zeta^n = z^m$.

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- Example 2 (§3.5). A branch of $\left(z^2 - 1\right)^{\frac{1}{2}}$ that is analytic outside the unit circle. Use $z\left(1 - \frac{1}{z^2}\right)^{\frac{1}{2}}$. See Maple worksheet.

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