Week 6: §§7.3 -7.4

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Möbius Transformation: Definition

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- f is 1-1 on its domain of definition and the inverse f^{-1} is again a Möbius transformation and is given by

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

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- But we may as well use the matrix obtained by removing the factor $(ad-bc)^{-1}$.

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- The above-mentioned special Möbius transformations map lines onto lines and circles onto circles. They are all 1-1 on all of $\mathbb{C}$.
- In the extended complex plane they satisfy $f(\infty) = \infty$.

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- See Maple illustration, where it is also shown that

$$\left(S^{-1}R_tS\right)(z) = \frac{z\cos\frac{t}{2} + i\sin\frac{t}{2}}{zi\sin\frac{t}{2} + \cos\frac{t}{2}}$$

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- If Σ_+ and Σ_- are the upper and lower Riemann hemispheres, then $SfS^{-1}\Sigma_- = \Sigma_+$.

Möbius Transformation: General properties

- Let $f(z) = \frac{az+b}{cz+d}$. Suppose $c \neq 0$. Then f can be written

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- f maps generalized circles onto generalized circles.
- Since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ the Möbius transformation f is conformal everywhere except at $-\frac{d}{c}$.

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- Since the interior and the exterior of C both are connected, so are their images under f.
- Since $1+i$ belongs to the interior of C and is mapped to ∞ it follows that the image of the interior of C is the exterior of the circle $\left|z+\frac{5}{3}\right| = \frac{4}{3}$.

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- Since the interior and the exterior of C both are connected, so are their images under f.
- Since $f(1+2i) = 1$ lies in the right half-plane, the interior of C is mapped onto the right half-plane.

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- $f(z_2) = i$ implies that $A\frac{z_2-z_3}{z_2-z_1} = i$.
- In the book $z_1 = 1, z_3 = -1$ and $z_2 = -i$ and we first find $f(z) = \frac{z+1}{z-1}$. But $f(0) = -1$ so take instead $f(z) = -\frac{z+1}{z-1}$.

Möbius Transformation: The Cross-Ratio (§7.4)

- Pick 3 arbitrary, but different points z_1, z_2, z_3 in the extended complex plane and require $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$.

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- Thus f is given by the *cross-ratio*

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- If one of the 3 points z_1, z_2, z_3 is ∞ then the cross-ratio is found by simply removing the factors in the numerator and denominator involving ∞.

Möbius Transformation a la carte (§7.4)

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Möbius Transformation a la carte (§7.4)

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- Examples 1 and 2 can be found in the Maple worksheet for Chapter 7.