# Week 8: §§4.4a-4.5
# Cauchy's Integral Theorem and Formula

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# Cauchy-Goursat Theorem for a Disk

- Theorem 9. If $f$ is analytic in the open disk $D$ then $f$ has an antiderivative in $D$ and for any closed contour $\Gamma$ in $D$

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- Lemma 1. If $R \subset D$ is a rectangle with sides parallel to the axes then $\int_R f(z)\,dz = 0$.
- Lemma 2. If $\int_R f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes then $f$ has an antiderivative in $D$.

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- Lemma 1. If $R \subset D$ is a rectangle with sides parallel to the axes then $\int_R f(z)\,dz = 0$.
- Lemma 2. If $\int_R f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes then $f$ has an antiderivative in $D$.
- By the Equivalence Theorem we conclude that loop integrals of $f$ in $D$ are zero.

# Proof of Lemma 1, I

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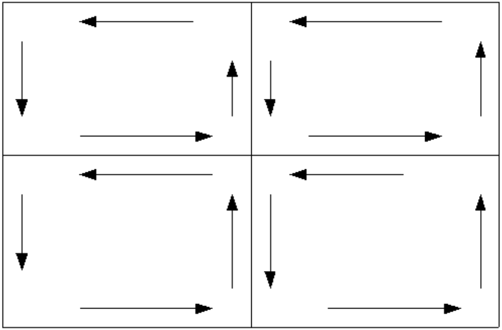

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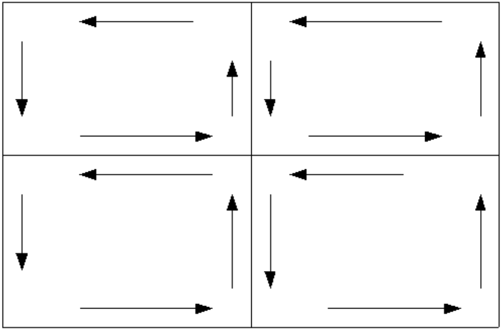

- $\int_R f(z)\,dz = \int_{R_1} f(z)\,dz + \int_{R_2} f(z)\,dz + \int_{R_3} f(z)\,dz + \int_{R_4} f(z)\,dz.$

# Proof of Lemma 1, II

- Since $\left|\int_R f(z)\,dz\right| \le \left|\int_{R_1} f(z)\,dz\right| + \left|\int_{R_2} f(z)\,dz\right| + \left|\int_{R_3} f(z)\,dz\right| + \left|\int_{R_4} f(z)\,dz\right|$ at least one of the subrectangles $R_k$ satisfies

$$\left|\int_{R_k} f(z)\,dz\right| \ge \frac{1}{4}\left|\int_R f(z)\,dz\right|$$

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- Select such a subrectangle, name it $R^{(1)}$, and divide it in four pieces as above: $R_k^{(1)}, k = 1, 2, 3, 4$. Then for (at least) one of those (named $R^{(2)}$) we have

$$\left|\int_{R^{(2)}} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{R^{(1)}} f(z)\,dz\right| \geq \frac{1}{4^2}\left|\int_R f(z)\,dz\right|$$

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- Continuing in this way we obtain a sequence of rectangles $R, R^{(1)}, R^{(2)}, \ldots, R^{(n)}, \ldots$ satisfying

$$\left|\int_{R^{(n)}} f(z)\,dz\right| \geq \frac{1}{4^n}\left|\int_R f(z)\,dz\right|$$

# Proof of Lemma 1, III

- If $P$ and $d$ are the perimeter and diagonal of $R$, respectively, then the perimeter and diagonal of $R^{(n)}$ are $2^{-n}P$ and $2^{-n}d$, respectively.

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- If $z \in R^{(n)}$ then $|z - z_0| \leq 2^{-n}d$.
- Since $f$ is differentiable at $z_0$ there exists for given $\varepsilon > 0$ a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

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- Pick $n$ such that $2^{-n}d \leq \delta$. Then for $z \in R^{(n)}$ we have

$$\left| f(z) - f(z_0) - f'(z_0)(z - z_0) \right| < \varepsilon |z - z_0| \leq \varepsilon 2^{-n} d$$

# Proof of Lemma 1, IV

- Since $\int_{R^{(n)}} 1 dz = 0$ and $\int_{R^{(n)}} (z - z_0)\, dz = 0$ we get

$$\begin{aligned}\left|\int_{R^{(n)}} f(z)\, dz\right| &= \left|\int_{R^{(n)}} \left(f(z) - f(z_0) - f'(z_0)(z - z_0)\right) dz\right| \\ &\le \max_{z\in R^{(n)}} \left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \ell\left(R^{(n)}\right) \\ &\le \varepsilon d 2^{-n} P 2^{-n} = \varepsilon d P 4^{-n}\end{aligned}$$

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- Thus

$$\left|\int_R f(z)\,dz\right| \leq 4^n \left|\int_{R^{(n)}} f(z)\,dz\right| \leq \varepsilon d P$$

- This inequality holds for every $\varepsilon > 0$ so $\int_R f(z)\,dz = 0$.

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- Lemma 2. Let $D$ be an open disk. If $\int_R f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes then $f$ has an antiderivative in $D$.

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- For any $z_1, z_2 \in D$ let $hv(z_1, z_2)$ be the path from $z_1$ to $z_2$ composed of first a horizontal line segment then a vertical line segment.

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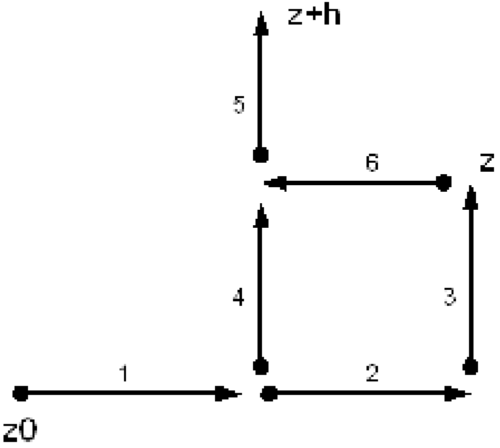
- We shall show that $F'(z) = f(z)$ for any $z \in D$, which means showing that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

# Proof of Lemma 2, II

- $F(z+h) - F(z) = \left( \int_{\gamma_1+\gamma_4+\gamma_5} - \int_{\gamma_1+\gamma_2+\gamma_3} \right) f(\zeta)\, d\zeta.$



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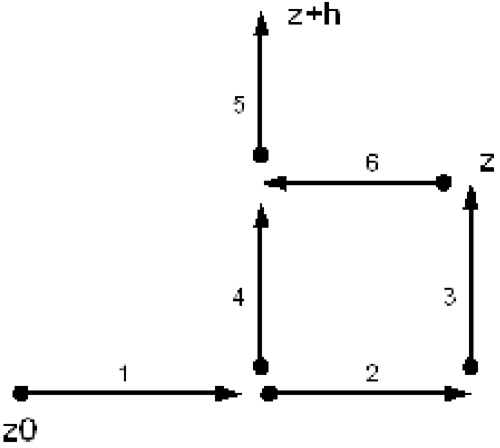


- $\int_{\gamma_2+\gamma_3+\gamma_6-\gamma_4} f(\zeta)\, d\zeta = 0$ so $F(z+h) - F(z) = \int_{\gamma_5+\gamma_6} f(\zeta)\, d\zeta = \int_{hv(z,z+h)} f(\zeta)\, d\zeta.$

# Proof of Lemma 2, III

- Since $F(z+h) - F(z) = \int_{hv(z,z+h)} f(\zeta)\, d\zeta$ we get

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{hv(z,z+h)} (f(\zeta) - f(z))\, d\zeta$$

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$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \varepsilon \ell (hv(z, z+h)) \leq 2\varepsilon$$

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- This completes the proof of Lemma 2 and the Cauchy-Goursat Theorem for a Disk.

# Cauchy's Integral Formula for Circle in Disk I

- Theorem. Let $f$ be analytic in the open disk $D$. Let $C$ be a circle contained in a square inscribed in $D$ (traversed counterclockwise). Then for any $z_0$ inside $C$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

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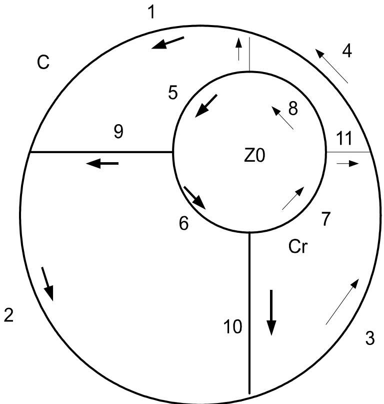
- The requirement that $C$ be inside a square inscribed in $D$ makes the proof easier.
- We get a more general version later anyway.
- Proof. Let $C_r$ be the positively oriented circle with center $z_0$ and radius $r > 0$ small enough to ensure that $C_r$ is inside $C$.

# Proof I

- $\frac{f(z)}{z-z_0}$ is analytic in each of 4 disks containing a piece of pineapple. Integrals around them are zero. So the sum of these is zero. Contributions from straight line segments cancel.



# Proof II

- Thus $\int_C \frac{f(z)}{z-z_0} dz = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} = \int_{\gamma_5+\gamma_6+\gamma_7+\gamma_8} = \int_{C_r} \frac{f(z)}{z-z_0} dz$.

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- So $\int_C \frac{f(z)}{z-z_0}dz = \int_{C_r} \frac{f(z)}{z-z_0}dz = \int_{C_r} \frac{f(z_0)}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz$.

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- It follows from the equation above that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz$ is independent of $r$.

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- So $\int_C \frac{f(z)}{z-z_0} dz = \int_{C_r} \frac{f(z)}{z-z_0} dz = \int_{C_r} \frac{f(z_0)}{z-z_0} dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz$.
- It follows from the equation above that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz$ is independent of $r$.
- We shall show that the value is zero.

# Proof III

- Since $f$ is differentiable at $z_0$ we have $\frac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)$ as $z \to z_0$ and as a consequence there exist an $r_0 > 0$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leq |f'(z_0)| + 1$$

as long as $|z - z_0| \leq r_0$.

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- Thus for $r \leq r_0$ we have

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \left( \left| f'(z_0) \right| + 1 \right) 2\pi r$$

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$$\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz\right| \leq (|f'(z_0)| + 1)\, 2\pi r$$

- Being independent of $r$ it follows that $\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz = 0$.

# A Differentiability Lemma

- Theorem 15. Let $g$ be continuous on the contour $\Gamma$. Define
$$G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$
for each $z$ not on $\Gamma$. Then $G$ is analytic at each $z$ not on $\Gamma$ and
$$G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$$

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- Theorem 15 generalized. Let $g$ be continuous on the contour $\Gamma$. Define $G$ as above. Then $G^{(k)}$ is analytic for all $k \geq 0$ at each $z$ not on $\Gamma$ and

$$G^{(k)}(z) = k! \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

# Proof

- We have

$$\frac{G(z+h)-G(z)}{h}=\int_\Gamma \frac{g(\zeta)}{(\zeta-z-h)(\zeta-z)}d\zeta$$

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$$\frac{1}{(\zeta-z-h)(\zeta-z)}-\frac{1}{(\zeta-z)^2}=\frac{h}{(\zeta-z-h)(\zeta-z)^2}$$
- Let $M$ be an upper bound for $|g|$ on $\Gamma$ and $d>0$ a lower bound for the distance from $z$ to $\Gamma$. For $|h|\leq\frac{1}{2}d$ we have $|\zeta-z-h|\geq|\zeta-z|-|h|\geq\frac{1}{2}d$ so
$$\left|\frac{g(\zeta)}{(\zeta-z-h)(\zeta-z)^2}\right|\leq\frac{M}{\frac{1}{2}d^3}$$

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$$\left|\frac{g(\zeta)}{(\zeta-z-h)(\zeta-z)^2}\right|\leq\frac{M}{\frac{1}{2}d^3}$$
- Thus
$$\left|\frac{G(z+h)-G(z)}{h}-\int_\Gamma\frac{g(\zeta)}{(\zeta-z)^2}d\zeta\right|\leq|h|\frac{2M}{d^3}\ell(\Gamma)$$

# Theorem 16

- If $f$ is analytic in the open disk $D$ and $C \subset D$ is a positively oriented circle then for any $z$ inside $C$

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- Theorem 16. Let $f$ be analytic in the domain $D$. Then derivatives of all orders exist and are analytic in $D$.

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- $\Gamma_0$ and $\Gamma_1$ are also instead said to be *homotopic* in $D$.
- See Maple examples for illustrations.
- Definition. If every loop in a domain $D$ is homotopic to a point in $D$ then $D$ is said to be *simply connected*.

# Deformation Invariance Theorem

- Theorem 8. Let $f$ be analytic in the domain $D$. Then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz$$

for any two loops $\Gamma_0$ and $\Gamma_1$ that are homotopic in $D$.

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- Example. Let $\Gamma$ be any simple closed curve traversed in the counter-clockwise direction and having 0 in its interior. Then

$$\int_\Gamma \frac{1}{z}dz = \int_{\text{Unit circle}} \frac{1}{z}dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}dt = 2\pi i$$

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- Example. From Cauchy's theorem it follows that

$$\int_{|z|=2} \frac{\cos z}{(z-7)(z+3i)} dz = 0$$

# Cauchy's Integral Theorem: Examples

- Example. Let $\Gamma$ be any circle not passing through $a$. Then

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$$\frac{1+i}{z(z-1)(z-i)} = \frac{1-i}{z} + \frac{i}{z-1} - \frac{1}{z-i}$$

we find

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- If instead $\Gamma$ has $0$ and $i$ in its interior and $1$ in its exterior then

$$\int_\Gamma \frac{1+i}{z\,(z-1)\,(z-i)}dz = 2\pi i\,(1-i+0-1) = 2\pi$$

# Cauchy's Integral Formula

- Theorem 14. Let $\Gamma$ be a simple closed positively oriented contour contained in a simply connected domain $D$. Let $f$ be analytic in $D$. Then for any $z_0$ inside $\Gamma$
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- Proof. We may replace $\Gamma$ by the circle $C_r$ given by $|z - z_0| = r$ and inside $\Gamma$.
- By the Deformation Invariance Theorem and the local version of Cauchy's integral formula we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz$$

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- Example. If $\Gamma$ is the positively oriented circle $|z-3|=5$ then

$$\int_\Gamma \frac{e^{2z}}{z\,(z-5i)}dz = \int_\Gamma \frac{\left(\frac{e^{2z}}{z-5i}\right)}{z}dz = 2\pi i\frac{e^0}{-5i} = -\frac{2\pi}{5}$$

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- Thus $\int_{C_0}\left(\frac{\cos z}{z}+\frac{\cos z}{z-\pi}\right)dz + \int_{C_1}\left(\frac{\cos z}{z}+\frac{\cos z}{z-\pi}\right)dz = 0$.

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- Thus $\int_{C_0}\left(\frac{\cos z}{z}+\frac{\cos z}{z-\pi}\right)dz + \int_{C_1}\left(\frac{\cos z}{z}+\frac{\cos z}{z-\pi}\right)dz = 0$.
- If $C$ is the circle $\left|z-\frac{\pi}{2}\right| = \pi$ then by deformation: $\int_c\left(\frac{\cos z}{z}+\frac{\cos z}{z-\pi}\right)dz = 0$.