Week 9: §§4.5-4.6

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Cauchy's Integral Formula and Theorem 15

- *Cauchy's integral formula*. Let Γ be simple, closed, positively oriented, and contained in a simply connected domain D. Let f be analytic in D. Then

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- Let g be continuous on the contour Γ. Then $G(z) = \int_\Gamma \frac{g(\zeta)}{\zeta - z} d\zeta \ (z \notin \Gamma)$, is analytic and $G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$.

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- In fact $G^{(k)}$ is analytic for all $k \geq 0$ and $G^{(k)}(z) = k!\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}}d\zeta$ for each z not on Γ.

Theorems 16 and 17

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- Theorem 17. If $f = u + iv$ is analytic in a domain D, then $u, v \in C^{\infty}(D)$.

Morera's Theorem and The Generalized Cauchy Integral Formula

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- Theorem 19 (*Generalized Cauchy Integral Formula*). Let Γ be a simple closed positively oriented contour contained in a simply connected domain D. Let f be analytic in D. Then for any $n \in N$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \text{ for any } z \text{ inside } \Gamma$$

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- Written differently

$$\int_\Gamma \frac{f(z)}{(z - z_0)^m} dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0)$$

The Generalized Cauchy Integral Formula: Examples

- Example A. Let Γ be the circle $|z-2|=1$ (counterclockwise). Let Log be the principal branch of the logarithm. Then

$$\int_\Gamma \frac{\mathrm{Log}\,(z)}{(z-2)^3}dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\mathrm{Log}\,(z)\Big|_{z=2} = -\frac{\pi i}{4}$$

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- Example B. Let now Γ be the circle $|z|=1$ (counterclockwise). Then

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- The method of Example A could not be used. We had to resort to explicit calculation.

Cauchy Estimates

- Theorem 20. Let f be analytic inside and on the circle C_R given by $|z - z_0| = R$. Suppose $|f(z)| \leq M$ for all z on C_R. Then for all $n \in N$

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- Thus we get

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi} \left|\int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}$$

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we have for any z_0 and any R

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- Theorem 22. *The Fundamental Theorem of Algebra*. Every nonconstant polynomial has at least one zero.

The Fundamental Theorem of Algebra: Proof

- Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$. Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$.

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- Thus there is an $R > 0$ such that for $|z| \geq R$

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- By Liouville's theorem f is constant. Thus $n = 0$.

The Mean-value Property

- Let f be analytic inside and on the circle C_R given by $|z - z_0| = R$. By Cauchy's Integral Formula

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- C_R can be parametrized by $\zeta = z_0 + R\ e^{it}$, $t \in [0, 2\pi]$, so

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- Example C. Take $f = \exp$, $z_0 = 0$. We get the result

$$1 = e^0 = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(R\ e^{it}\right) dt$$

which is not entirely obvious, but is confirmed by Maple.

The Mean-value Property and the Maximum Modulus Principle

- Lemma 1. Let f be analytic in the disk B given by $|z - z_0| \leq R$. If $\max_{z \in B} |f(z)| = |f(z_0)|$ then $|f|$ is constant in B.

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- Proof. Since for any $r \leq R$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + r\, e^{it}\right) \right| dt$$

and since $\left| f\left(z_0 + r\, e^{it}\right) \right| \leq |f(z_0)|$ for all t, we in fact must have $\left| f\left(z_0 + r\, e^{it}\right) \right| = |f(z_0)|$ for all t and all $r \leq R$.

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- Theorem 23. *The Maximum Modulus Principle*. Let f be analytic in the domain D. Suppose $|f(z)|$ attains its maximum at a point $z_0 \in D$. Then f is constant.

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- Theorem 23. *The Maximum Modulus Principle*. Let f be analytic in the domain D. Suppose $|f(z)|$ attains its maximum at a point $z_0 \in D$. Then f is constant.
- Remark. Notice that the conclusion is that f is constant, not only $|f|$.

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- $|f|$ also attains its maximum at w. Thus by Lemma1 $|f|$ is constant in B. So necessarily $c_m = b$, and $|f(z_1)| = |f(z_0)|$.
- But if $|f|$ is constant in D, so is f (see the lecture notes for week 3).

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- Proof. As a continuous function $|f|$ attains its maximum at some point z_0 of the compact set $\overline{D}$. If $z_0 \in D$ then f is constant in D and by continuity also in $\overline{D}$.
- Theorem 26. If ϕ is harmonic in a simply connected domain D and attains its maximum or minimum at some point in D, then ϕ is constant.
- Proof. ϕ is the real part of an analytic function f. Suppose $z_0 \in D$ is a maximum point for ϕ. Then $\left|e^{f(z)}\right| = e^{\phi(z)}$ attains its maximum at z_0. However, $\exp \circ f$ is analytic, so must be constant in D. Therefore ϕ is constant.

The Maximum Modulus Principle II

- Theorem 24. *The Maximum Modulus Principle: Version 2.* Let f be analytic in the bounded domain D. Suppose f is continuous on the closure $\overline{D}$. Then f attains its maximum modulus on the boundary of D.
- Proof. As a continuous function $|f|$ attains its maximum at some point z_0 of the compact set $\overline{D}$. If $z_0 \in D$ then f is constant in D and by continuity also in $\overline{D}$.
- Theorem 26. If ϕ is harmonic in a simply connected domain D and attains its maximum or minimum at some point in D, then ϕ is constant.
- Proof. ϕ is the real part of an analytic function f. Suppose $z_0 \in D$ is a maximum point for ϕ. Then $\left|e^{f(z)}\right| = e^{\phi(z)}$ attains its maximum at z_0. However, $\exp \circ f$ is analytic, so must be constant in D. Therefore ϕ is constant.
- A minimum is handled by considering $-\phi$ instead.