

Week 12: §6.1 and §6.3

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November 27, 2008

The Residue

- ▶ Let f have an isolated singularity at z_0 . The coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion for f in an annulus $0 < |z - z_0| < R$ is the *residue of f at z_0* and is denoted by $\text{Res}(f; z_0)$ or $\text{Res}(z_0)$.

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- ▶ If Γ is a simple closed positively oriented contour having z_0 in its interior and lying in the annulus then
$$\int_{\Gamma} f(z) dz = 2\pi i a_{-1} = 2\pi i \text{Res}(z_0).$$

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- ▶ If Γ is a simple closed positively oriented contour having z_0 in its interior and lying in the annulus then
$$\int_{\Gamma} f(z) dz = 2\pi i a_{-1} = 2\pi i \text{Res}(z_0).$$
- ▶ **Example A.** Compute the integral $\oint_{|z|=1/2} \frac{-\text{Log}(1-z)}{z^4} dz$ (counterclockwise).

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- ▶ If Γ is a simple closed positively oriented contour having z_0 in its interior and lying in the annulus then $\int_{\Gamma} f(z) dz = 2\pi i a_{-1} = 2\pi i \text{Res}(z_0)$.
- ▶ Example A. Compute the integral $\oint_{|z|=1/2} \frac{-\text{Log}(1-z)}{z^4} dz$ (counterclockwise).
- ▶ The residue of the function $f(z) = \frac{-\text{Log}(1-z)}{z^4}$ at 0 is found from

$$\frac{-\text{Log}(1-z)}{z^4} = z^{-4} \sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{z^{n-4}}{n} = \sum_{n=-3}^{\infty} \frac{z^n}{n+4}$$

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- ▶ If Γ is a simple closed positively oriented contour having z_0 in its interior and lying in the annulus then $\int_{\Gamma} f(z) dz = 2\pi i a_{-1} = 2\pi i \text{Res}(z_0)$.
- ▶ Example A. Compute the integral $\oint_{|z|=1/2} \frac{-\text{Log}(1-z)}{z^4} dz$ (counterclockwise).
- ▶ The residue of the function $f(z) = \frac{-\text{Log}(1-z)}{z^4}$ at 0 is found from

$$\frac{-\text{Log}(1-z)}{z^4} = z^{-4} \sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{z^{n-4}}{n} = \sum_{n=-3}^{\infty} \frac{z^n}{n+4}$$

- ▶ We see that $\text{Res}(0) = \frac{1}{3}$ so that

$$\oint_{|z|=1/2} \frac{-\text{Log}(1-z)}{z^4} dz = 2\pi i \frac{1}{3} = \frac{2}{3}\pi i.$$

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- If f has a simple pole at z_0 so that $f(z) = \sum_{j=-1}^{\infty} a_j (z - z_0)^j$ then

$$\operatorname{Res}(z_0) = a_{-1} = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

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- ▶ Example 2 (in the book). Let f be the ratio of two functions h and g both analytic at z_0 , $f(z) = \frac{h(z)}{g(z)}$.

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- ▶ If g has a simple zero at z_0 and $h(z_0) \neq 0$ then

$$\operatorname{Res}(z_0) = a_{-1} = \lim_{z \rightarrow z_0} (z - z_0) \frac{h(z)}{g(z)} = \frac{h(z_0)}{g'(z_0)}$$

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- ▶ Example B. The singularities of $\tan z = \frac{\sin z}{\cos z}$ are $z = \frac{\pi}{2} + p\pi, p \in \mathbb{Z}$. They are all simple and $\sin\left(\frac{\pi}{2} + p\pi\right) = (-1)^p \neq 0$.

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- ▶ Since $\frac{d}{dz} \cos z = -\sin z$ we get for all $p \in \mathbb{Z}$

$$\operatorname{Res}\left(\frac{\pi}{2} + p\pi\right) = -\frac{\sin\left(\frac{\pi}{2} + p\pi\right)}{\sin\left(\frac{\pi}{2} + p\pi\right)} = -1$$

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- Theorem 1. If f has a pole of order m at z_0 , then

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

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- Example C. We find the residues of all singularities in the disk $|z| < 4$ of

$$f(z) = \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4}$$

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- ▶ Example C. We find the residues of all singularities in the disk $|z| < 4$ of

$$f(z) = \frac{\operatorname{Log}(4 - z)}{z^2 (z - 3)^4}$$

- ▶ f has a pole of order 2 at 0 and pole of order 3 at 3 (not order 4, since $\operatorname{Log}(4 - z)$ has a simple zero at 3) at 3.

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► $\operatorname{Res}(0) = \lim_{z \rightarrow 0} \frac{d}{dz} [z^2 f(z)] =$

$$\lim_{z \rightarrow 0} \frac{d}{dz} \left[\frac{\operatorname{Log}(4-z)}{(z-3)^4} \right] =$$

$$\lim_{z \rightarrow 0} \left[\frac{1}{(z-3)^4(z-4)} - 4 \frac{\operatorname{Log}(4-z)}{(z-3)^5} \right] = \frac{8}{243} \ln 2 - \frac{1}{324}.$$

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- ▶ $\operatorname{Res}(3) = \frac{1}{2} \lim_{z \rightarrow 3} \frac{d^2}{dz^2} [(z-3)^3 f(z)] =$

$$\frac{1}{2} \lim_{z \rightarrow 3} \frac{d^2}{dz^2} \left[\frac{\operatorname{Log}(4-z)}{z^2(z-4)} \right] = -\frac{1}{27} \text{ (used Maple for the last$$

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- **Theorem 2.** Let Γ be a simple closed positively oriented contour and suppose f is analytic inside and on Γ except at the points z_1, z_2, \dots, z_n inside Γ . Then

$$\int_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f; z_k)$$

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$$\int_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f; z_k)$$

- ▶ Proof. We may write

$$\int_{\Gamma} f(z) dz = \sum_{j=1}^n \int_{C_k} f(z) dz$$

where C_k is a positively oriented circle with center at z_k and a radius r_k chosen small enough to ensure that the disks $|z - z_k| \leq r_k$ don't intersect. See Maple animation.

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- ▶ In the disk $|z - z_k| \leq r_k$ we write $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_k)^j$.

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- ▶ In the disk $|z - z_k| \leq r_k$ we write

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_k)^j.$$

- ▶ Then $\int_{C_k} f(z) dz = 2\pi i a_{-1} = 2\pi i \text{Res}(f; z_k)$.

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- Let C be the circle $|z - 1 - i| = \frac{5}{2}$ traversed counterclockwise. We evaluate the integral

$$\oint_C \frac{z^2 + z + 1}{(z - 1)(z - 2i)(z + 2 - i)} dz$$

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- ▶ The integrand $f(z)$ is analytic inside C except at the two simple poles 1 and $2i$.

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Cauchy's Residue Theorem, Example D

- ▶ Let C be the circle $|z - 1 - i| = \frac{5}{2}$ traversed counterclockwise. We evaluate the integral

$$\oint_C \frac{z^2 + z + 1}{(z - 1)(z - 2i)(z + 2 - i)} dz$$

- ▶ The integrand $f(z)$ is analytic inside C except at the two simple poles 1 and $2i$.
- ▶ Thus $\oint_C f(z) dz = 2\pi i (\text{Res}(1) + \text{Res}(2i))$.

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$$\oint_C \frac{z^2 + z + 1}{(z - 1)(z - 2i)(z + 2 - i)} dz$$

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- ▶ Thus $\oint_C f(z) dz = 2\pi i (\text{Res}(1) + \text{Res}(2i))$.
- ▶ $\text{Res}(1) = \lim_{z \rightarrow 1} (z - 1) f(z) =$
 $\lim_{z \rightarrow 1} \frac{z^2 + z + 1}{(z - 2i)(z + 2 - i)} = \frac{3}{(1 - 2i)(3 - i)} = \frac{3}{50} + \frac{21}{50}i.$

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- ▶ Let C be the circle $|z - 1 - i| = \frac{5}{2}$ traversed counterclockwise. We evaluate the integral

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- ▶ Thus $\oint_C f(z) dz = 2\pi i (\text{Res}(1) + \text{Res}(2i))$.

- ▶ $\text{Res}(1) = \lim_{z \rightarrow 1} (z - 1) f(z) =$
 $\lim_{z \rightarrow 1} \frac{z^2 + z + 1}{(z - 2i)(z + 2 - i)} = \frac{3}{(1 - 2i)(3 - i)} = \frac{3}{50} + \frac{21}{50}i$.

- ▶ $\text{Res}(2i) = \lim_{z \rightarrow 2i} (z - 2i) f(z) =$
 $\lim_{z \rightarrow 2i} \frac{z^2 + z + 1}{(z - 1)(z + 2 - i)} = \frac{-3 + 2i}{(-1 + 2i)(2 + i)} = \frac{18}{25} + \frac{1}{25}i$.

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- ▶ Let C be the circle $|z - 1 - i| = \frac{5}{2}$ traversed counterclockwise. We evaluate the integral

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- ▶ The integrand $f(z)$ is analytic inside C except at the two simple poles 1 and $2i$.

- ▶ Thus $\oint_C f(z) dz = 2\pi i (\text{Res}(1) + \text{Res}(2i))$.

- ▶ $\text{Res}(1) = \lim_{z \rightarrow 1} (z - 1) f(z) =$
 $\lim_{z \rightarrow 1} \frac{z^2 + z + 1}{(z - 2i)(z + 2 - i)} = \frac{3}{(1 - 2i)(3 - i)} = \frac{3}{50} + \frac{21}{50}i$.

- ▶ $\text{Res}(2i) = \lim_{z \rightarrow 2i} (z - 2i) f(z) =$
 $\lim_{z \rightarrow 2i} \frac{z^2 + z + 1}{(z - 1)(z + 2 - i)} = \frac{-3 + 2i}{(-1 + 2i)(2 + i)} = \frac{18}{25} + \frac{1}{25}i$.

- ▶ Thus $\oint_C f(z) dz = 2\pi i \left(\frac{3}{50} + \frac{21}{50}i + \frac{18}{25} + \frac{1}{25}i \right) =$
 $\left(-\frac{23}{25} + \frac{39}{25}i \right) \pi$.

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- We evaluate the integral

$$\oint_{|z|=2} \left(\tan z + \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4} \right) dz$$

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- ▶ We evaluate the integral

$$\oint_{|z|=2} \left(\tan z + \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4} \right) dz$$

- ▶ First we write the integral as the sum

$$\oint_{|z|=2} \tan z dz + \oint_{|z|=2} \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4} dz$$

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- ▶ We evaluate the integral

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- ▶ First we write the integral as the sum

$$\oint_{|z|=2} \tan z dz + \oint_{|z|=2} \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4} dz$$

- ▶ Then we use residues to find

$$2\pi i \left(\operatorname{Res} \left(\tan z; -\frac{\pi}{2} \right) + \operatorname{Res} \left(\tan z; \frac{\pi}{2} \right) + \operatorname{Res} \left(\frac{\operatorname{Log}(4-z)}{z^2(z-3)^4}; 0 \right) \right)$$

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- ▶ We evaluate the integral

$$\oint_{|z|=2} \left(\tan z + \frac{\operatorname{Log}(4-z)}{z^2(z-3)^4} \right) dz$$

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- ▶ Then we use residues to find

$$2\pi i \left(\operatorname{Res} \left(\tan z; -\frac{\pi}{2} \right) + \operatorname{Res} \left(\tan z; \frac{\pi}{2} \right) + \operatorname{Res} \left(\frac{\operatorname{Log}(4-z)}{z^2(z-3)^4}; 0 \right) \right)$$

- ▶ But these residues we found earlier, so the given integral has the value

$$2\pi i \left(-1 + (-1) + \frac{8}{243} \ln 2 - \frac{1}{324} \right) = 2\pi i \left(\frac{8}{243} \ln 2 - \frac{649}{324} \right)$$

Integrals over \mathbb{R} by Residues I

- Suppose the limit

$$\lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} f(x) dx$$

exists, then its value is called the *Cauchy principal value* of the integral of f over \mathbb{R} and is denoted by

$$p.v. \int_{-\infty}^{\infty} f(x) dx$$

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Integrals over \mathbb{R} by Residues I

- Suppose the limit

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- If both of the integrals

$$\int_0^{\infty} f(x) dx \text{ and } \int_{-\infty}^0 f(x) dx$$

exist, then the Cauchy principle value does too and

$$p.v. \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx + \int_{-\infty}^0 f(x) dx$$

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- ▶ Suppose the limit

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$$p.v. \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx + \int_{-\infty}^0 f(x) dx$$

- ▶ The converse is false. See next slide.

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- Example. We have $p.v. \int_{-\infty}^{\infty} \sin x \, dx = 0$, but neither $\int_0^{\infty} \sin x \, dx$ nor $\int_{-\infty}^0 \sin x \, dx$ exists.

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- ▶ Example. We have $p.v. \int_{-\infty}^{\infty} \sin x \, dx = 0$, but neither $\int_0^{\infty} \sin x \, dx$ nor $\int_{-\infty}^0 \sin x \, dx$ exists.
- ▶ Example. $p.v. \int_{-\infty}^{\infty} \cos x \, dx$ does not exist: $\int_{-\rho}^{\rho} \cos x \, dx = 2 \sin \rho$. No limit as $\rho \rightarrow \infty$.

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- ▶ Example. We have $p.v. \int_{-\infty}^{\infty} \sin x \, dx = 0$, but neither $\int_0^{\infty} \sin x \, dx$ nor $\int_{-\infty}^0 \sin x \, dx$ exists.
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- ▶ Example F. We find $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6+64}$, which clearly is convergent in the usual sense.

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- ▶ Example F. We find $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6+64}$, which clearly is convergent in the usual sense.
- ▶ Consider $f(z) = \frac{z^4}{z^6+64}$ in \mathbb{C} . It has the 6 simple poles $\pm\sqrt{3} \pm i$ (all 4 combinations) and $\pm 2i$.

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- ▶ Example. We have $p.v. \int_{-\infty}^{\infty} \sin x \, dx = 0$, but neither $\int_0^{\infty} \sin x \, dx$ nor $\int_{-\infty}^0 \sin x \, dx$ exists.
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- ▶ Consider $f(z) = \frac{z^4}{z^6+64}$ in \mathbb{C} . It has the 6 simple poles $\pm\sqrt{3} \pm i$ (all 4 combinations) and $\pm 2i$.
- ▶ Let Γ_{ρ} consist of the real line segment $[-\rho, \rho]$ and the circular arc C_{ρ}^+ parametrized by $z = \rho e^{it}$, $t \in [0, \pi]$.

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- ▶ Example. We have $p.v. \int_{-\infty}^{\infty} \sin x \, dx = 0$, but neither $\int_0^{\infty} \sin x \, dx$ nor $\int_{-\infty}^0 \sin x \, dx$ exists.
- ▶ Example. $p.v. \int_{-\infty}^{\infty} \cos x \, dx$ does not exist: $\int_{-\rho}^{\rho} \cos x \, dx = 2 \sin \rho$. No limit as $\rho \rightarrow \infty$.
- ▶ Example F. We find $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6+64}$, which clearly is convergent in the usual sense.
- ▶ Consider $f(z) = \frac{z^4}{z^6+64}$ in \mathbb{C} . It has the 6 simple poles $\pm\sqrt{3} \pm i$ (all 4 combinations) and $\pm 2i$.
- ▶ Let Γ_{ρ} consist of the real line segment $[-\rho, \rho]$ and the circular arc C_{ρ}^+ parametrized by $z = \rho e^{it}$, $t \in [0, \pi]$.
- ▶ For $\rho > 2$ the poles in the upper half-plane are inside Γ_{ρ} . Thus

$$\int_{\Gamma_{\rho}} f(z) \, dz = 2\pi i \left(\operatorname{Res}(2i) + \operatorname{Res}(\sqrt{3} + i) + \operatorname{Res}(-\sqrt{3} + i) \right)$$

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- ▶ The residues are easily calculated, so for $\rho > 2$

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \left(-\frac{i}{12} + \frac{\sqrt{3}-i}{24} + \frac{-\sqrt{3}-i}{24} \right) = \frac{1}{3}\pi$$

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Integrals over \mathbb{R} by Residues III

- ▶ The residues are easily calculated, so for $\rho > 2$

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \left(-\frac{i}{12} + \frac{\sqrt{3}-i}{24} + \frac{-\sqrt{3}-i}{24} \right) = \frac{1}{3}\pi$$

- ▶ Now we show that the contribution from the circular arc tends to zero as $\rho \rightarrow \infty$.

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Integrals over \mathbb{R} by Residues III

- ▶ The residues are easily calculated, so for $\rho > 2$

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \left(-\frac{i}{12} + \frac{\sqrt{3}-i}{24} + \frac{-\sqrt{3}-i}{24} \right) = \frac{1}{3}\pi$$

- ▶ Now we show that the contribution from the circular arc tends to zero as $\rho \rightarrow \infty$.
- ▶ We have

$$\begin{aligned} \left| \int_{C_\rho^+} f(z) dz \right| &= \left| \int_0^\pi \frac{\rho^4 e^{4it} i \rho e^{it}}{\rho^6 e^{6it} + 64} dt \right| \\ &\leq \int_0^\pi \frac{\rho^5}{\rho^6 - 64} dt = \pi \frac{\rho^5}{\rho^6 - 64} \\ &= \pi \frac{1}{\rho - \frac{64}{\rho^5}} \rightarrow 0 \text{ as } \rho \rightarrow \infty \end{aligned}$$

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- ▶ The residues are easily calculated, so for $\rho > 2$

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \left(-\frac{i}{12} + \frac{\sqrt{3}-i}{24} + \frac{-\sqrt{3}-i}{24} \right) = \frac{1}{3}\pi$$

- ▶ Now we show that the contribution from the circular arc tends to zero as $\rho \rightarrow \infty$.
- ▶ We have

$$\begin{aligned} \left| \int_{C_\rho^+} f(z) dz \right| &= \left| \int_0^\pi \frac{\rho^4 e^{4it} i \rho e^{it}}{\rho^6 e^{6it} + 64} dt \right| \\ &\leq \int_0^\pi \frac{\rho^5}{\rho^6 - 64} dt = \pi \frac{\rho^5}{\rho^6 - 64} \\ &= \pi \frac{1}{\rho - \frac{64}{\rho^5}} \rightarrow 0 \text{ as } \rho \rightarrow \infty \end{aligned}$$

- ▶ We conclude that $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6+64} = \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{x^4 dx}{x^6+64} = \frac{\pi}{3}$.

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- Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho^+} f(z) dz = 0$$

where C_ρ^+ is defined as in Example F.

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- Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho^+} f(z) dz = 0$$

where C_ρ^+ is defined as in Example F.

- Example G. Consider $\int_{-\infty}^{\infty} \frac{3x^5+x^4}{x^6+64} dx$. The improper integral is not convergent in the usual sense, but the Cauchy principal value exists:

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- ▶ Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

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- ▶ Example G. Consider $\int_{-\infty}^{\infty} \frac{3x^5 + x^4}{x^6 + 64} dx$. The improper integral is not convergent in the usual sense, but the Cauchy principal value exists:
- ▶ We have

$$\begin{aligned} \text{p.v.} \int_{-\infty}^{\infty} \frac{3x^5 + x^4}{x^6 + 64} dx &= \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{3x^5 + x^4}{x^6 + 64} dx \\ &= \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{3x^5}{x^6 + 64} dx + \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{x^4}{x^6 + 64} dx \end{aligned}$$

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- ▶ Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

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- ▶ $\int_{-\infty}^{\infty} \frac{3x^5}{x^6 + 64} dx = 0$ since $\frac{3x^5}{x^6 + 64}$ is an odd function of x .

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- ▶ Lemma 1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with $\deg Q \geq \deg P + 2$ then

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho^+} f(z) dz = 0$$

where C_ρ^+ is defined as in Example F.

- ▶ Example G. Consider $\int_{-\infty}^{\infty} \frac{3x^5 + x^4}{x^6 + 64} dx$. The improper integral is not convergent in the usual sense, but the Cauchy principal value exists:
- ▶ We have

$$\begin{aligned} p.v. \int_{-\infty}^{\infty} \frac{3x^5 + x^4}{x^6 + 64} dx &= \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{3x^5 + x^4}{x^6 + 64} dx \\ &= \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{3x^5}{x^6 + 64} dx + \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{x^4}{x^6 + 64} dx \end{aligned}$$

- ▶ $\int_{-\infty}^{\infty} \frac{3x^5}{x^6 + 64} dx = 0$ since $\frac{3x^5}{x^6 + 64}$ is an odd function of x .
- ▶ Thus $p.v. \int_{-\infty}^{\infty} \frac{3x^5 + x^4}{x^6 + 64} dx = \frac{\pi}{3}$.

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► We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.

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- ▶ We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.
- ▶ Since $0 \leq \frac{e^{ax}}{1+e^x} \leq e^{(a-1)x}$ the improper integral $\int_0^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent.

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- ▶ We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.
- ▶ Since $0 \leq \frac{e^{ax}}{1+e^x} \leq e^{(a-1)x}$ the improper integral $\int_0^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent.
- ▶ Since $0 \leq \frac{e^{ax}}{1+e^x} \leq e^{ax}$ the improper integral $\int_{-\infty}^0 \frac{e^{ax}}{1+e^x} dx$ is convergent.

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- ▶ We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.
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- ▶ Thus certainly $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent when $0 < a < 1$.

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- ▶ We find $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ when $0 < a < 1$.
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- ▶ Thus certainly $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ is convergent when $0 < a < 1$.
- ▶ Let Γ_ρ be the contour consisting of γ_1 :
 $z = x, x \in [-\rho, \rho]$, γ_2 : $z = \rho + iy, y \in [0, 2\pi]$, γ_3 :
 $z = x + 2\pi i, x \in [-\rho, \rho]$ (but opposite), γ_4 :
 $z = -\rho + iy, y \in [0, 2\pi]$ (again opposite).

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 $z = x + 2\pi i, x \in [-\rho, \rho]$ (but opposite), γ_4 :
 $z = -\rho + iy, y \in [0, 2\pi]$ (again opposite).
- ▶ $f(z) = \frac{e^{az}}{1+e^z}$ has the simple pole πi inside the contour.

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 $z = x + 2\pi i, x \in [-\rho, \rho]$ (but opposite), γ_4 :
 $z = -\rho + iy, y \in [0, 2\pi]$ (again opposite).
- ▶ $f(z) = \frac{e^{az}}{1+e^z}$ has the simple pole πi inside the contour.

▶ Thus

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \operatorname{Res}(\pi i) = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i}.$$

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▶ Thus

$$\int_{\Gamma_\rho} f(z) dz = 2\pi i \operatorname{Res}(\pi i) = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i}.$$

- ▶ We have $\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(\rho+iy)}}{1+e^{\rho+iy}} i dy \right| \leq$
 $\int_0^{2\pi} \frac{e^{a\rho}}{e^\rho - 1} dy = 2\pi \frac{e^{a\rho}}{e^\rho - 1} \rightarrow 0$ as $\rho \rightarrow \infty$.

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► We have $\left| \int_{\gamma_4} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(-\rho+iy)}}{1+e^{-\rho+iy}} idy \right| \leq$
 $\int_0^{2\pi} \frac{e^{-a\rho}}{1-e^{-\rho}} dy = 2\pi \frac{e^{-a\rho}}{1-e^{-\rho}} \rightarrow 0$ as $\rho \rightarrow \infty$.

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▶ We have $\left| \int_{\gamma_4} f(z) dz \right| = \left| \int_0^{2\pi} \frac{e^{a(-\rho+iy)}}{1+e^{-\rho+iy}} i dy \right| \leq$
 $\int_0^{2\pi} \frac{e^{-a\rho}}{1-e^{-\rho}} dy = 2\pi \frac{e^{-a\rho}}{1-e^{-\rho}} \rightarrow 0$ as $\rho \rightarrow \infty$.

▶ Now

$$\begin{aligned} \int_{\gamma_3} f(z) dz &= - \int_{-\rho}^{\rho} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx = -e^{a2\pi i} \int_{-\rho}^{\rho} \frac{e^{ax}}{1+e^x} dx \\ &= -e^{a2\pi i} \int_{\gamma_1} f(z) dz \end{aligned}$$

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 $\int_0^{2\pi} \frac{e^{-a\rho}}{1-e^{-\rho}} dy = 2\pi \frac{e^{-a\rho}}{1-e^{-\rho}} \rightarrow 0$ as $\rho \rightarrow \infty$.

► Now

$$\begin{aligned} \int_{\gamma_3} f(z) dz &= - \int_{-\rho}^{\rho} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx = -e^{a2\pi i} \int_{-\rho}^{\rho} \frac{e^{ax}}{1+e^x} dx \\ &= -e^{a2\pi i} \int_{\gamma_1} f(z) dz \end{aligned}$$

► Thus

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{\rho \rightarrow \infty} \int_{\gamma_1} f(z) dz = \frac{-2\pi i e^{a\pi i}}{1 - e^{a2\pi i}} = \frac{\pi}{\sin(a\pi)}$$