Solutions to Problem Session 1

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Solution 1 (A) *Part 3.*

$$\begin{aligned}\left(\sqrt{3}-i\right)^m &= (1+i)^n \Leftrightarrow 2^m \exp\left(-i\frac{\pi}{6}m\right) = \left(\sqrt{2}\right)^n \exp\left(i\frac{\pi}{4}n\right)\\ &\Leftrightarrow 2^m = \left(\sqrt{2}\right)^n \wedge -\frac{\pi}{6}m = \frac{\pi}{4}n + p2\pi \text{ for some } p \in Z\\ &\Leftrightarrow n = 2m \wedge -\frac{1}{6}m = \frac{1}{4}2m + p2 \text{ for some } p \in Z\\ &\Leftrightarrow n = 2m \wedge m = -3p \text{ for some } p \in Z\end{aligned}$$

Thus the equation $\left(\sqrt{3}-i\right)^m = (1+i)^n$ *is satisfied iff* m *is divisible by* 3 *and* n *is twice as large as* m*.*

Solution 2 (B) *See the Maple solution.*

Solution 3 (C) *We have when* $|z| = 2$

$$\begin{aligned}\left|z^3 - z\right| &= \left|z\left(z^2-1\right)\right| = |z|\left|z^2-1\right| = 2\left|z^2-1\right|\\ &\leq 2\left(\left|z^2\right|+1\right) = 2\left(|z|^2+1\right) = 10\end{aligned}$$

and also

$$\left|z^3 - z\right| = 2\left|z^2-1\right| \geq 2\left(\left|z^2\right|-1\right) = 2\left(|z|^2-1\right) = 6$$

We have equality in the inequality $\left|z^2-1\right| \leq \left|z^2\right|+1$ *when the arguments of* z^2 *and* -1 *are the same. Thus e.g. when* $\arg\left(z^2\right) = \pi$*. But this is the case when* $\arg z = \frac{\pi}{2}$*. Thus we have equality for* $z = 2i$*. Therefore* $M = 10$ *is optimal, i.e. with* $f(z) = z^3 - z$ *we have* $|f(2i)| = 10$*.*
Similarly we find $z = 2$ *realizes the minimum:* $|f(2)| = 6$*, so that* $m = 6$ *is optimal.*
When $|z| = 2$ *is replaced by* $|z| = \frac{1}{2}$ *we find similarly that* $M = \frac{5}{8}, m = \frac{3}{8}$*. Both are optimal since* $\left|f\left(\frac{1}{2}\right)\right| = \frac{3}{8}$ *and* $\left|f\left(\frac{i}{2}\right)\right| = \frac{5}{8}$*.*

Solution 4 (D) *Part 2. Let* $z = x + iy, \quad x, y \in R$ *and let* $w = re^{i\theta}$ *with* $r \geq 0, \theta \in R$*. Then*

$$e^z = w \Leftrightarrow e^x e^{iy} = re^{i\theta} \Leftrightarrow e^x = r \wedge y = \theta + p2\pi$$

for some $p \in Z$. *Thus* $e^z = w$ *can be solved iff* $e^x = r$ *can be solved. The latter can be solved iff* $r > 0$. *Thus* $e^z = w$ *can be solved iff* $w \neq 0$.
Part 3. We solve $e^z = ie$. *We do as above:*

$$\begin{aligned} e^z &= ie \Leftrightarrow e^x e^{iy} = e e^{i\frac{\pi}{2}} \Leftrightarrow e^x = e \wedge y = \frac{\pi}{2} + p2\pi \\ &\Leftrightarrow x = 1 \wedge y = \frac{\pi}{2} + p2\pi \end{aligned}$$

for some $p \in Z$. *So the solutions are* $z = 1 + i\left(\frac{\pi}{2} + p2\pi\right), \quad p \in Z$.
We solve $e^z = i(1-e) = -i(e-1)$. *As before*

$$\begin{aligned} e^z &= i(1-e) \Leftrightarrow e^x e^{iy} = (e-1)e^{-i\frac{\pi}{2}} \Leftrightarrow e^x = e-1 \wedge y = -\frac{\pi}{2} + p2\pi \\ &\Leftrightarrow x = \ln(e-1) \wedge y = -\frac{\pi}{2} + p2\pi \end{aligned}$$

for some $p \in Z$. *So the solutions are* $z = \ln(e-1) + i\left(-\frac{\pi}{2} + p2\pi\right), \quad p \in Z$.

Solution 5 (E) *We have* $z(t) = \frac{t-i}{t+i}, \quad t \in R$. *Here we only consider part 4. Let us consider an arbitrary point on the unit circle* $w = e^{i\theta}$. *Then*

$$z(t) = w \Leftrightarrow \frac{t-i}{t+i} = e^{i\theta} \Leftrightarrow t\left(1 - e^{i\theta}\right) = i\left(1 + e^{i\theta}\right)$$

When $e^{i\theta} = 1$ *the left hand side is zero regardless of the value of* t, *whereas the right hand side is* $2i$. *Thus the point* 1 *on the unit circle is not part of the curve described by* z. *If* $e^{i\theta} \neq 1$ *we find*

$$t = \frac{i\left(1 + e^{i\theta}\right)}{1 - e^{i\theta}} = \frac{i\left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right)}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} \in R$$

We conclude that all other points on the unit circle belong to the curve described by z.

Solution 6 (F) *Suppose that we have an ordering relation satisfying the properties stated. The imaginary unit* i *must either satisfy* $i > 0$ *or* $-i > 0$.
Suppose $i > 0$. *Then* $i^2 > 0$. *Thus* $-1 > 0$ *from which follows* $(-1)^2 > 0$, *i.e.* $1 > 0$. *But one of the requirements was that if* $x > 0$ *then* $-x < 0$. *Therefore we have reached contradiction, so we cannot have* $i > 0$. *The only other possibility* $-i > 0$ *is ruled out by the same argument.*