Solutions to Problem Session 5

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1 Problem session

Solution 1 (A) *Consider the real functions* $u_j : \mathbb{R}^2 \to \mathbb{R},\ j = 1, 2,$ *given by*

$$u_1(x, y) = 2x(1 - y) \quad \text{and} \quad u_2(x, y) = x^3 - 3xy^2$$

1. *We show that these functions are harmonic by applying the definition:*

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} &= \frac{\partial^2}{\partial x^2}(2x(1-y)) + \frac{\partial^2}{\partial y^2}(2x(1-y)) = 0 + 0 = 0 \\
\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= \frac{\partial^2}{\partial x^2}\left(x^3 - 3xy^2\right) + \frac{\partial^2}{\partial y^2}\left(x^3 - 3xy^2\right) = 6x - 6x = 0
\end{aligned}$$

2. *We determine for each of the functions* u_j *all harmonic conjugates* v_j *so that* $f_j = u_j + iv_j$ *is an analytic function. We use the Cauchy-Riemann equations:*

$$\begin{aligned}
v_y &= u_x \\
v_x &= -u_y
\end{aligned}$$

For $u = u_1 = 2x(1 - y)$ *we find*

$$v_y = u_x = 2 - 2y$$

from which follows $v = 2y - y^2 + h(x)$. *This implies* $v_x = h'(x)$. *However, we also have* $v_x = -u_y = 2x$, *so* $h'(x) = 2x$ *and consequently* $h(x) = x^2 + C$. *Therefore the harmonic conjugates of* u_1 *are given by*

$$v_1(x, y) = x^2 - y^2 + 2y + C$$

where $C \in R$. *We have*

$$\begin{aligned}
f_1(z) &= f_1(x + iy) = u_1(x, y) + iv_1(x, y) = 2x(1 - y) + i\left(x^2 - y^2 + 2y + C\right) \\
&= i(x + iy)^2 + 2(x + iy) + iC = iz^2 + 2z + iC
\end{aligned}$$

For $u = u_2 = x^3 - 3xy^2$ *we find*

$$v_y = u_x = 3x^2 - 3y^2$$

from which follows $v = 3x^2y - y^3 + h(x)$. *This implies* $v_x = 6xy + h'(x)$. *However, we also have* $v_x = -u_y = 6xy$, *so* $h'(x) = 0$ *and consequently* $h(x) = C$. *Therefore the harmonic conjugates of* u_2 *are given by*

$$v_2(x, y) = 3x^2y - y^3 + C$$

where $C \in R$. We have

$$\begin{aligned} f_2(z) &= f_2(x+iy) = u_2(x,y) + iv_2(x,y) = x^3 - 3xy^2 + i\left(3x^2y - y^3 + C\right) \\ &= (x+iy)^3 + iC = z^3 + iC \end{aligned}$$

Solution 2 (B) *Consider any branch of the logarithm of the form*

$$\mathcal{L}_\tau(z) = \ln|z| + i\arg_\tau(z), \quad \textit{where } \arg_\tau(z) \in]\tau, \tau + 2\pi].$$

$v(z) = \arg_\tau(z)$ is a harmonic function in the slit plane $D_\tau^* = \mathbb{C} \setminus \{re^{i\tau} \mid r \geq 0\}$, because $\mathcal{L}_\tau(z)$ is analytic in D_τ^*. It is not harmonic in $\mathbb{C} \setminus \{0\}$ for the simple reason that it is not continuous there.
Let D be a domain given as the union of two domains $D = D_1 \cup D_2$. If $u : D \to \mathbb{R}$ is harmonic in D_1 and in D_2, then u is harmonic in D, since to be harmonic in some set means to satisfy Laplace's differential equation $\Delta u = 0$ in each point of the set. This simple fact implies that $u(z) = \ln|z|$ is a harmonic function in $\mathbb{C} \setminus \{0\}$ it is the real part of both $Log\,(z)$ and $\mathcal{L}_0(z)$.

Solution 3 (C) *We shall find a function u which is harmonic in the region sketched in fig. 3.12. We use*

$$\phi(x,y) = A\arg_{-\frac{\pi}{4}}(z-1-i) + B$$

since $\arg_{-\frac{\pi}{4}}(z-1-i)$ is the imaginary part of $\mathcal{L}_{-\frac{\pi}{4}}(z-1-i)$ which is analytic in the region. Imposing the boundary condtions gives

$$\begin{aligned} 10 &= \phi(x,1) = A \cdot 0 + B,\ x > 1 \\ 0 &= \phi(1,y) = A \cdot \left(\frac{3\pi}{2}\right) + B,\ y < 1 \end{aligned}$$

Thus $B = 10$ and $A = -\frac{20}{3\pi}$ so

$$\phi(x,y) = -\frac{20}{3\pi}\arg_{-\frac{\pi}{4}}(z-1-i) + 10$$

The value at $(0,0)$ is $\phi(0,0) = -\frac{20}{3\pi}\arg_{-\frac{\pi}{4}}(-1-i) + 10 = -\frac{20}{3\pi}\frac{5\pi}{4} + 10 = \frac{5}{3}$.

Solution 4 (D) *1. Consider the first Example on p. A-22:*

$$w = f(z) = z^2$$

and the domain, the half strip,

$$D = \{\ z = x + iy \mid 0 < x < 1 \wedge y > 0\ \}$$

The domain D is bounded by the three curves

$$\gamma_1\ :\ z = iy,\ y \geq 0\ ; \quad \gamma_2\ :\ z = x,\ 0 \leq x \leq 1\ ; \quad \gamma_3\ :\ z = 1 + iy,\ y \geq 0$$

We find

$$\begin{aligned} f(iy) &= -y^2 \textit{ so } f(\gamma_1) = R_- \cup \{0\} \\ f(x) &= x^2 \textit{ so } f(\gamma_2) = [0,1] \\ f(1+iy) &= (1+iy)^2 = 1 - y^2 + 2iy \end{aligned}$$

so $f(\gamma_3)$ is the upper branch of the parabola $u = 1 - \frac{1}{4}v^2$ in the uv-plane.
That f is conformal in D follows from $f'(z) = 2z \neq 0$ in D. That f maps D injectively, hence bijectively onto $E = f(D)$ follows from

$$z_1^2 = z_2^2 \Leftrightarrow z_1 = \pm z_2$$

and from the fact that not both of $\pm z_2$ can belong to D.
The inverse function $f^{-1} : E \to D$ is a square root, and it is the principal branch of the square root, since D belongs to the right half plane.

2. *Consider the last Example on p. A-22:*

$$w = f(z) = e^z$$

and the domain, the strip,

$$D = \{ \ z \mid 0 < \operatorname{Im} z < \pi \ \}$$

f is conformal in D since the derivative is clearly different from zero. f maps D injectively, hence bijectively onto the upper half plane $E = f(D)$, since it is injective in the larger strip $\{ \ z \mid -\pi < \operatorname{Im} z \leq \pi \ \}$. The inverse function $f^{-1} : E \to D$ is the well-known function Log.

Solution 5 (E) *We consider the sine function in the first Example on p. A-24.*

1. *The vertical half-lines which bound the half strip S_+ given as*

$$S_+ = \{ \ z = x + iy \mid |x| < \tfrac{\pi}{2} \ \wedge y > 0 \ \}$$

are given by $z = \pm\frac{\pi}{2} + iy, y > 0$. We find

$$\begin{aligned} \sin\left(\frac{\pi}{2} + iy\right) &= \frac{1}{2i}\left(\exp\left(i\frac{\pi}{2} - y\right) - \exp\left(-i\frac{\pi}{2} + y\right)\right) \\ &= \frac{1}{2i}\left(ie^{-y} + ie^{y}\right) = \cosh y \end{aligned}$$

and also

$$\begin{aligned} \sin\left(-\frac{\pi}{2} + iy\right) &= \frac{1}{2i}\left(\exp\left(-i\frac{\pi}{2} - y\right) - \exp\left(i\frac{\pi}{2} + y\right)\right) \\ &= \frac{1}{2i}\left(-ie^{-y} - ie^{y}\right) = -\cosh y \end{aligned}$$

Thus the vertical half-line to the right is mapped onto $]1, \infty[$ and the vertical half-line to the left is mapped onto $]-\infty, -1[$.
The sine function is conformal in S_+ since its derivative cos *is different from zero in S_+. We have*

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

Since the imaginary part is positive $\sin(S_+)$ is at least a part of the upper half-plane. Keeping $y > 0$ fixed and letting $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ vary we see that $u + iv = \sin(x + iy)$ takes on all values on the upper half of the ellipse

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

When y increases from 0 and up the ellipse expands from being just the real line segment $]-1, 1[$ to being larger and larger having semi-axes $\cosh y$ and $\sinh y$. Thus we see that $\sin(S_+)$ is the upper half-plane.

2. *By using that* $\overline{\exp z} = \exp(\overline{z})$ *we get*

$$\overline{\sin z} = \overline{\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)} = -\frac{1}{2i}\left(e^{-i\overline{z}} - e^{i\overline{z}}\right) = \sin \overline{z}$$

3. *By using* $\sin \overline{z} = \overline{\sin z}$ *we see that the lower strip* $S_- = \{\ z = x + iy \mid |x| < \frac{\pi}{2}\ \wedge y < 0\ \}$ *is mapped bijectively onto the lower half-plane.*