Solutions to Problem Session 9

Preben Alsholm

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1 Problem session

Solution 1 (A) ***Cauchy's Generalized Integral Formulas and the Deformation Invariance Theorem.***

1. Let Γ denote the square loop with vertices at the points $\pm(2 \pm 2i)$ oriented counterclockwise. By Cauchy's Generalized Integral Formula we get for any integer $n \geq 0$

$$\int_\Gamma \frac{e^{-z}}{(z - i\frac{\pi}{2})^{n+1}}\, dz = \frac{2\pi i}{n!} \left(\frac{d^n}{dz^n} \left(e^{-z}\right) \right) \Bigg|_{z=i\frac{\pi}{2}} = \frac{2\pi}{n!} (-1)^n$$

2. Since $\pm 3i$ are outside the circle (counterclockwise!) we find

$$\int_{|z|=2} \frac{\cos z}{z(z^2+9)}\, dz = 2\pi i \frac{\cos 0}{9} = \frac{2\pi i}{9}$$

For the other integral we find by deformation invariance

$$\int_{|z-3i|=4} \frac{\cos z}{z(z^2+9)}\, dz = \int_{|z|=2} \frac{\cos z}{z(z^2+9)}\, dz + \int_{|z-3i|=1} \frac{\cos z}{z(z^2+9)}\, dz$$

Thus since $z^2 + 9 = (z - 3i)(z + 3i)$

$$\int_{|z-3i|=4} \frac{\cos z}{z(z^2+9)}\, dz = \frac{2\pi i}{9} + 2\pi i \frac{\cos(3i)}{3i\,(6i)} = \frac{2\pi i}{9} - \pi i \frac{\cosh 3}{9}$$

Solution 2 (B) ***Application of Liouville's Theorem.***
Let f be an entire function. Assume there is a positive constant M such that $\operatorname{Re} f(z) \leq M$ for all z in $\mathbb{C}$, hence that the real part of f is bounded from above. Show that f is constant.
We use the hint and consider the function defined as $F(z) = e^{f(z)}$. This function satisfies

$$|F(z)| = \left| e^{f(z)} \right| = e^{\operatorname{Re} f(z)} \leq e^M$$

for all z in $\mathbb{C}$. By Liouville's Theorem F is constant. Therefore $F'(z) = 0$ so $f'(z) = 0$ for all z. Thus f is constant.

Solution 3 (C) ***The Maximum Modulus Principle.***
Given the functions

$$f(z) = e^{z^2} \quad \text{and} \quad g(z) = z^2 - \tfrac{1}{4}$$

1. The maximum modulus of f and the maximum modulus of g in the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ are attained on the boundary. Here we find

$$\left|f\left(e^{i\theta}\right)\right| = e^{\cos(2\theta)}$$

which has maximal value e. For g we have

$$\left|g\left(e^{i\theta}\right)\right| = \left|e^{2i\theta} - \frac{1}{4}\right|$$

which has maximal value $\frac{5}{4}$.

2. Set

$$F(z) = \frac{1}{f(z)} \quad \text{and} \quad G(z) = \frac{1}{g(z)}$$

G is not analytic in $\mathbb{D}$, thus the maximum modulus principle cannot be applied to G, and in fact it has no maximal modulus in $\overline{\mathbb{D}}$. However, for $F(z) = e^{-z^2}$ we find the maximal modulus to be the maximum of

$$e^{-\cos(2\theta)}$$

which is e (again). Thus the minimum modulus for f is e^{-1}.

Solution 4 (D) ***Application of Cauchy's Estimates.***
Let f be an entire function. Assume there is a positive constant K such that $|f(z)| \leq K|z|$ for all $z \in \mathbb{C}$.

1. *Show that $f''(z) = 0$ for all $z \in \mathbb{C}$. We use the hint and consider for an arbitrary point z_0 in $\mathbb{C}$, the circle $C_R : \ |z - z_0| = R$. We have on C_R*

$$|f(z)| \leq K|z| \leq K\left(|z_0| + R\right)$$

from which follows

$$\left|f''\left(z_0\right)\right| \leq \frac{2K\left(|z_0| + R\right)}{R^2}$$

By letting R tend to $+\infty$ we see that $\left|f''\left(z_0\right)\right| = 0$.

2. *Since $f''(z) = 0$ for all $z \in \mathbb{C}$ we have $f'(z) = a$ for some constant a, and $f(z) = az + b$ for some constant b. By $|f(z)| \leq K|z|$ we find that $f(0) = 0$ so we conclude that $f(z) = az$.*

3. *Let g be an entire function. Assume there is a positive constant L such that $|g(z)| \leq L|z|^2$ for all sufficiently large values of $|z|$, i.e. for all $|z| \geq r_0$ for a suitable $r_0 > 0$.*
Since $|g(z)|$ is bounded on the disk $|z| \leq r_0$ we have a constant M such that

$$|g(z)| \leq M\left(|z|^2 + 1\right)$$

for all $z \in \mathbb{C}$. Again consider for an arbitrary point z_0 in $\mathbb{C}$, the circle $C_R : \ |z - z_0| = R$. We have on C_R

$$|g(z)| \leq M\left(|z|^2 + 1\right) \leq M\left(\left(|z_0| + R\right)^2 + 1\right)$$

from which follows

$$|g'''(z_0)| \leq \frac{3!M\left(\left(|z_0| + R\right)^2 + 1\right)}{R^3}$$

By letting R tend to $+\infty$ we see that $\left|g'''\left(z_0\right)\right| = 0$. Thus $g(z) = az^2 + bz + c$.

4. *In general we have: Let g be an entire function. Assume there is a positive constant L such that $|g(z)| \leq L|z|^n$ (where $n \in N$) for all sufficiently large values of $|z|$, i.e. for all $|z| \geq r_0$ for a suitable $r_0 > 0$. Then $g(z)$ is a polynomial of degree at most n.*
   The proof is just like above.

Solution 5 (E) ***Application of the Fundamental Theorem of Algebra.***
Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

denote a polynomial of degree n, and let $z_1, z_2, \ldots, z_r$ denote the different roots of multiplicity $d_1, d_2, \ldots, d_r$ respectively. Hence

$$P(z) = a_n (z - z_1)^{d_1} (z - z_2)^{d_2} \cdots (z - z_r)^{d_r}$$

1. *We have*

$$\begin{aligned} \frac{P'(z)}{P(z)} &= \frac{\sum_{j=1}^r d_j a_n (z - z_1)^{d_1} (z - z_2)^{d_2} \cdots (z - z_2)^{d_j - 1} \cdots (z - z_r)^{d_r}}{a_n (z - z_1)^{d_1} (z - z_2)^{d_2} \cdots (z - z_r)^{d_r}} \\ &= \sum_{j=1}^r \frac{d_j}{z - z_j} = \frac{d_1}{z - z_1} + \frac{d_2}{z - z_2} + \cdots + \frac{d_r}{z - z_r} \end{aligned}$$

2. *Let Γ be a simple closed positively oriented curve, not passing through any of the roots of P. Then*

$$\begin{aligned} \int_\Gamma \frac{P'(z)}{P(z)} \, dz &= \int_\Gamma \sum_{j=1}^r \frac{d_j}{z - z_j} \, dz = \sum_{j=1}^r \int_\Gamma \frac{d_j}{z - z_j} \, dz \\ &= \sum_{z_j \ inside \ \Gamma} \int_\Gamma \frac{d_j}{z - z_j} \, dz = 2\pi i \sum_{z_j \ inside \ \Gamma} d_j \end{aligned}$$

   Thus

$$\frac{1}{2\pi i} \int_\Gamma \frac{P'(z)}{P(z)} \, dz \ = \ \textit{the number of roots inside } \Gamma \textit{ counted with multiplicity.}$$

Solution 6 (F) ***Estimates of absolute values of rational functions for $|z|$ sufficiently large.***
Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

denote a polynomial of degree n. For $z \neq 0$

$$P(z) = z^n \left(a_n + a_{n-1}/z + \cdots + a_1/z^{n-1} + a_0/z^n \right)$$

1. *Setting $K = |a_n| + |a_{n-1}| + \cdots + |a_0|$ we find for $|z| \geq 1$*

$$\begin{aligned} |P(z)| &= |z|^n \left| \left(a_n + a_{n-1}/z + \cdots + a_1/z^{n-1} + a_0/z^n \right) \right| \\ &\leq |z|^n \left| \Big(|a_n| + |a_{n-1}| / |z| + \cdots + |a_1| / |z|^{n-1} + |a_0| / |z|^n \Big) \right| \\ &\leq |z|^n \left(|a_n| + |a_{n-1}| + \cdots + |a_0| \right) = K|z|^n \end{aligned}$$

2. *We have as* $z \to \infty$

$$\frac{P(z)}{z^n} \to a_n$$

Thus for R *sufficiently large we have*

$$\frac{|P(z)|}{|z|^n} \;\geq\; \frac{|a_n|}{2} \quad \textit{for} \quad |z| \geq R$$

3. *Let* $f(z) = P(z)/Q(z)$ *where* Q *is a polynomial of degree* m*. Then there is a constant* L *such that*

$$|Q(z)| \geq L\,|z|^m$$

for all sufficiently large $|z|$*. Therefore for all sufficiently large* $|z|$

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{K\,|z|^n}{L\,|z|^m} = \frac{K}{L\,|z|^{m-n}}$$