Solutions to Problem Session 10

Preben Alsholm

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1 Problem session

Solution 1 (A) ***Geometric series.***

1. The series
$$\sum_{j=0}^{+\infty}(z-1)^j$$
is a geometric series. It is convergent for $|z-1|<1$ and divergent otherwise. For $|z-1|<1$ the sum of the series is
$$\sum_{j=0}^{+\infty}(z-1)^j = \frac{1}{1-(z-1)} = \frac{1}{2-z}$$

2. The series
$$\sum_{j=0}^{+\infty}\frac{1}{(z-1)^j} \quad \text{and} \quad \sum_{j=1}^{+\infty}\frac{1}{(z-1)^j}$$
are both convergent for $|z-1|>1$ and divergent otherwise. For $|z-1|>1$ the sums of the series are
$$\sum_{j=0}^{+\infty}\frac{1}{(z-1)^j} = \frac{1}{1-\left(\frac{1}{z-1}\right)} = \frac{z-1}{z-2}$$
$$\sum_{j=1}^{+\infty}\frac{1}{(z-1)^j} = \sum_{j=0}^{+\infty}\frac{1}{(z-1)^j}-1 = \frac{z-1}{z-2}-1 = \frac{1}{z-2}$$

3. The series
$$\sum_{j=0}^{+\infty}e^{jz}$$
is again a geometric series and is convergent for $|e^z| = e^{\operatorname{Re} z} < 1$ and divergent otherwise. When convergent the sum is
$$\sum_{j=0}^{+\infty}e^{jz} = \frac{1}{1-e^z}$$

4. The series

$$\sum_{j=0}^{+\infty}\left(\frac{z-1}{z+1}\right)^j$$

is a geometric series and is convergent for $\left|\frac{z-1}{z+1}\right| < 1$ and divergent otherwise. The domain of convergence is therefore given by $\operatorname{Re} z > 0$. The sum is

$$\sum_{j=0}^{+\infty}\left(\frac{z-1}{z+1}\right)^j = \frac{1}{1-\frac{z-1}{z+1}} = \frac{1}{2}(z+1)$$

Solution 2 (B) ***Domains of convergence.***

We determine for each of the following series the maximal domain in which the series is absolutely convergent.

1. $\sum_{j=1}^{+\infty}\frac{1}{j!\ z^j}$. *We recognize the Taylor expansion of* $\exp\left(z^{-1}\right)$ *(except for the first term). Thus the series is absolutely convergent for all* $z \neq 0$.

2. $\sum_{j=1}^{+\infty}\frac{j}{j+1}(2z)^j$. *Using the ratio test we find*

$$\left|\frac{\frac{j+1}{j+2}(2z)^{j+1}}{\frac{j}{j+1}(2z)^j}\right| = 2\frac{(j+1)^2}{j(j+2)}|z| = 2\frac{(1+1/j)^2}{(1+2/j)}|z| \to 2|z|$$

as $j\to\infty$. *Thus the series is absolutely convergent for* $|z| < \frac{1}{2}$ *and divergent for* $|z| > \frac{1}{2}$. *The radius of convergence is* $\rho = \frac{1}{2}$.

3. $\sum_{j=1}^{+\infty} j!\left(\frac{z}{j}\right)^j$. *Again we use the ratio test:*

$$\left|\frac{(j+1)!\left(\frac{z}{j+1}\right)^{j+1}}{j!\left(\frac{z}{j}\right)^j}\right| = |z|\left(\frac{j}{j+1}\right)^j = |z|\frac{1}{\left(1+\frac{1}{j}\right)^j} \to |z|\,e^{-1}$$

as $j\to\infty$. *Thus the series is absolutely convergent for* $|z| < e$ *and divergent for* $|z| > e$. *The radius of convergence is* $\rho = e$.

Solution 3 (C) ***Taylor series.***

Consider the function

$$f(z) = \frac{1}{3-z}$$

1. We find the Maclaurin series of f. We use the hint given and rewrite

$$\frac{1}{3-z} = \frac{1}{3}\frac{1}{1-z/3} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z}{3}\right)^j = \sum_{j=0}^{\infty}3^{-j-1}z^j$$

Thus the coefficients a_j of the Maclaurin series are given by

$$a_j = 3^{-j-1}$$

The radius of convergence is 3 (geometric series!).

2. We find the Taylor series of f around $z_0 = 1$ and use the hint.

$$\begin{aligned} \frac{1}{3-z} &= \frac{1}{2-(z-1)} = \frac{1}{2}\frac{1}{1-\frac{z-1}{2}} = \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z-1}{2}\right)^j \\ &= \sum_{j=0}^{\infty} 2^{-j-1}(z-1)^j \end{aligned}$$

Thus the coefficients a_j of the Taylor series are given by

$$a_j = 2^{-j-1}$$

The radius of convergence is 2 (geometric series).

Solution 4 (D) ***Cauchy Multiplication.***
Consider the functions

$$g(z) = \frac{z}{1+e^z} \quad \textit{and} \quad h(z) = \frac{\sin z}{1+z^2}$$

1. We determine the four coefficients $a_j,\ j = 0, 1, 2, 3$, in the Maclaurin series of g, which we write as

$$g(z) = \sum_{j=0}^{\infty} a_j z^j$$

$\sum_{j=0}^{\infty} a_j z^j$We have

$$(1+e^z)\, g(z) = z$$

so

$$\left(1 + \sum_{j=0}^{\infty}\frac{z^j}{j!}\right)\sum_{j=0}^{\infty} a_j z^j = z$$

Thus

$$\left(2 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots\right)\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) = z$$

Comparing powers we get

$$\begin{aligned} 2a_0 &= 0 \\ 2a_1 + a_0 &= 1 \\ 2a_2 + a_1 + \frac{1}{2}a_0 &= 0 \\ 2a_3 + a_2 + \frac{1}{2}a_1 + \frac{1}{3!}a_0 &= 0 \end{aligned}$$

Thus $a_0 = 0, a_1 = \frac{1}{2}, a_2 = -\frac{1}{4}, a_3 = 0$.

2. We determine the Maclaurin series of h. Writing

$$h(z) = \sum_{j=0}^{\infty} a_j z^j$$

we find from

$$\left(1+z^2\right) h(z) = \sin z$$

that

$$\left(1+z^2\right)\sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

But

$$\begin{aligned}\left(1+z^2\right)\sum_{j=0}^{\infty} a_j z^j &= \sum_{j=0}^{\infty} a_j z^j + \sum_{j=2}^{\infty} a_{j-2} z^j \\ &= a_0 + a_1 z + \sum_{j=2}^{\infty} \left(a_{j-2} + a_j\right) z^j\end{aligned}$$

Therefore

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_{2k-2} + a_{2k} &= 0, k \geq 1 \\ a_{2k-1} + a_{2k+1} &= \frac{(-1)^k}{(2k+1)!}, k \geq 1 \end{aligned}$$

Therefore

$$\begin{aligned} a_{2k} &= 0, k > 0 \\ a_{2k+1} &= (-1)^k \left(1 + \frac{1}{3!} + \frac{1}{5!} + \cdots + \frac{1}{(2k+1)!}\right) \end{aligned}$$

The series for $h(z)$ is

$$h(z) = \sum_{j=0}^{\infty} (-1)^j \left(\sum_{q=0}^{j} \frac{1}{(2q+1)!}\right) z^{2j+1}$$

Solution 5 (E) ***Uniform convergence.***
Consider the geometric series

$$\sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j}$$

1. The series is convergent for $\left|\frac{3}{z-i}\right| < 1$ and divergent otherwise. Thus the domain of convergence is given by $|z-i| > 3$.
2. The geometric series is uniformly convergent on any set given by $\left|\frac{3}{z-i}\right| \leq r$ where $r < 1$, i.e. on the closed region $|z-i| \geq \frac{3}{r}$. But the circle $|z| = 10$ is included in such a region if $\frac{3}{r} > 4$, i.e. for $r < \frac{3}{4}$.
3. By the uniform convergence we may interchange integration and summation so

$$\int_{|z|=10} \left(\sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j}\right) dz = \sum_{j=0}^{+\infty} \int_{|z|=10} \frac{3^j}{(z-i)^j} dz$$

All terms but one are zero so

$$\int_{|z|=10} \left(\sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j} \right) dz \ = \int_{|z|=10} \frac{3}{z-i} \, dz = 6\pi i$$