Solutions to Problem Session 11

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1 Problem session

Solution 1 (A) ***Laurent series, obtained from geometric series.***
Consider as in ***Exercise C****, the 10th problem sheet, the function*

$$f(z) = \frac{1}{3-z}$$

1. *We determine the Laurent series for f in the annulus $|z| > 3$ and use the hint given. We get*

$$\begin{aligned} \frac{1}{3-z} &= -\frac{1}{z}\frac{1}{1-3/z} = -\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{3}{z}\right)^j \\ &= -\sum_{j=0}^{\infty} 3^j z^{-j-1} = -\sum_{j=-\infty}^{-1} 3^{-j-1} z^j \end{aligned}$$

for $|z| > 3$. Thus $a_j = 0$ for $j \geq 0$ and $a_j = -3^{-j-1}$ for $j \leq -1$.

2. *We determine the Laurent series for f in the annulus $|z-1| > 2$. Again we rewrite*

$$\begin{aligned} \frac{1}{3-z} &= \frac{1}{2-(z-1)} = -\frac{1}{z-1}\frac{1}{1-\frac{2}{z-1}} = -\frac{1}{z-1}\sum_{j=0}^{\infty}\left(\frac{2}{z-1}\right)^j \\ &= -\sum_{j=0}^{\infty} 2^j (z-1)^{-j-1} = -\sum_{j=-\infty}^{-1} 2^{-j-1} (z-1)^j \end{aligned}$$

for $|z-1| > 2$. Thus $a_j = 0$ for $j \geq 0$ and $a_j = -2^{-j-1}$ for $j \leq -1$.

3. *The Laurent series for f in the annulus $|z-3| > 0$ is easily found. In fact there is nothing to do. The coefficients: $a_{-1} = -1$ and all other are zero.*

Solution 2 (B) ***Laurent series, obtained from known Maclaurin series.***
Given

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right) \quad \text{and} \quad g(z) = \frac{e^z}{(z+1)^2}$$

1. *We determine the Laurent series of f in the annulus $|z| > 0$. We use that*

$$\sin z = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

with radius of convergence ∞. Thus for $z \neq 0$

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right) = z^2 \sum_{j=0}^{\infty} (-1)^j \frac{\left(z^{-2}\right)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{-4j}}{(2j+1)!}$$

Therefore $a_{-4j} = \frac{(-1)^j}{(2j+1)!}$ for all $j \geq 0$. All other are zero.

2. *We determine the Laurent series of g in the annulus $|z+1| > 0$ and use the hint. We have for all z*

$$e^{z+1} = \sum_{j=0}^{\infty} \frac{(z+1)^j}{j!}$$

Thus we have for all $z \neq -1$

$$g(z) = \frac{e^z}{(z+1)^2} = e^{-1} \sum_{j=0}^{\infty} \frac{(z+1)^{j-2}}{j!} = e^{-1} \sum_{j=-2}^{\infty} \frac{(z+1)^j}{(j+2)!}$$

The coefficients: $a_j = \frac{e^{-1}}{(j+2)!}$ for $j \geq -2$, all other zero.

3. *Let C denote the circle $|z| = 3$. By integrating termwise we find*

$$\begin{aligned}
\int_C f(z)\, dz &= \int_C \sum_{j=0}^{\infty} (-1)^j \frac{z^{-4j}}{(2j+1)!} dz = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \int_C z^{-4j} dz = 0 \\
\int_C g(z)\, dz &= \int_C e^{-1} \sum_{j=-2}^{\infty} \frac{(z+1)^j}{(j+2)!} dz = \sum_{j=-2}^{\infty} e^{-1} \frac{1}{(j+2)!} \int_C (z+1)^j\, dz \\
&= 2\pi i e^{-1}
\end{aligned}$$

Solution 3 (C) ***Isolated singularities.***
Determine in each of the cases below whether z_0 is an isolated singularity of f, and if so state its type: removable, pole of order m, or essential. For a pole the order must be given. In case of a removable singularity determine also $\lim_{z \to z_0} f(z)$.

1. $f(z) = \frac{iz+1}{z-2}$ *and* $z_0 = 2$. *Isolated singularity, pole of order 1.*
2. $f(z) = \frac{iz+1}{z-2}$ *and* $z_0 = \infty$. *Since $f(z) \to i$ as $z \to \infty$ we have a removable singularity at ∞.*
3. $f(z) = \frac{z}{(z^2+1)^3}$ *and* $z_0 = i$. *Isolated singularity, pole of order 3.*
4. $f(z) = \frac{\cos z - 1}{z^2}$ *and* $z_0 = 0$. *Isolated singularity, removable.* $\lim_{z \to 0} f(z) = -\frac{1}{6}$.
5. $f(z) = \sin\left(\frac{1}{z}\right)$ *and* $z_0 = 0$. *Isolated essential singularity.*
6. $f(z) = \sin z$ *and* $z_0 = \infty$. *By the result above we have an isolated essential singularity.*
7. $f(z) = \frac{e^z - 1}{z^2}$ *and* $z_0 = 0$. *Isolated singularity, pole of order 1.*
8. $f(z) = \frac{1}{\sin(1/z)}$ *and* $z_0 = 0$. *The singularity is not isolated since the sequence of singularities $\left(\frac{1}{p\pi}\right)_{p=1}^{\infty}$ converges to zero.*

Solution 4 (D) ***Cauchy Multiplication of Laurent series.***
Given $f(z) = \cot z$.

1. *We determine all zeros and all isolated singularities of* f*. The zeros are the zeros os* $\cos z$*, i.e.* $z = \frac{\pi}{2} + p\pi, p \in \mathbb{Z}$*. The singularities of* f *are the zeros of* $\sin z$*, i.e.* $z = p\pi, p \in \mathbb{Z}$*. These are all simple poles.*
2. *The function* f *has infinitely many Laurent expansions around* $z = 0$ *in infinitely many different annuli of the form* $A_{R_1,R_2} = \{ z \in \mathbb{C} \mid R_1 < |z| < R_2 \}$*. However, in each of the annuli* A_{R_1,R_2} *where* f *is analytic the expansion is unique.*
The largest possible annulus of the form $0 < |z| < R$ *is* $0 < |z| < \pi$*. We determine the part* $a_{-1}z^{-1} + a_0 + a_1 z$ *of the Laurent expansion of* f *in this annulus by using the known expansions for* sin *and* cos*. We have*
$$\cos z = \sin z \cdot \left(a_{-1}z^{-1} + a_0 + a_1 z + \cdots\right)$$
Thus
$$\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} \cdot \left(a_{-1}z^{-1} + a_0 + a_1 z + \cdots\right)$$
and
$$1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots = \left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right)\left(a_{-1}z^{-1} + a_0 + a_1 z + \cdots\right)$$
so
$$1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots = a_{-1} + a_0 z + \left(a_1 - \frac{1}{3!}a_{-1}\right)z^2 + \cdots$$
which implies that $a_{-1} = 1, a_0 = 0, a_1 = -\frac{1}{3}$.

Solution 5 (E) ***Radius of convergence.***

1. *The radius of convergence of the Maclaurin series of the function*
$$f(z) = \frac{4z^2 - \pi^2}{\cos z}$$
is $\frac{3\pi}{2}$ *since the singularities at* $\pm\frac{\pi}{2}$ *are removable* f *. But* f *has singularities at* $\pm\frac{3\pi}{2}$.
2. *The radius of convergence of the Taylor series around* $z_0 = -1 - i$ *of the principle branch of the logarithm Log is* $|-1 - i| = \sqrt{2}$.

Solution 6 (F) ***An incorrect argument.***
What is wrong with the following argument:
The claim
$$\sum_{j=0}^{+\infty} z^j = \frac{1}{1-z}$$
is valid only for $|z| < 1$ *and*
$$\sum_{j=1}^{+\infty} \frac{1}{z^j} = \frac{1}{z-1}$$
is valid only for $|z| > 1$*. Thus we cannot add the two series for any* $z \in \mathbb{C}$.