Complex Analysis 01141

Department of Mathematics

Week 1, 2008

1 Coverage

In the **1st week** we cover §§ 1.1 − 1.4. Basic calculations with complex numbers and the complex exponential are reviewed. In a course named *Complex Analysis* it is clearly impossible to succeed without complete mastery of the basics. It is not enough that your computer or calculator can do the computations. You yourself should be able to do them too.

In the **2nd week** we cover §§ 1.5 − 1.7 og 2.1 − 2.2. *Multiple-valued* expressions for complex roots and powers, *the extended complex plane*, the so-called *Riemann sphere*, and limits and continuity.

2 Comments on the material for this and the coming week

Exact expressions If the exact answer is $\sqrt{3}$ or $\frac{\pi}{3}$ then that should be given and not approximated values like 1.732050808 or 1.047197551. It is useful to recall that for instance

$$e^{i\pi/4} = \tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}, \quad e^{i\pi/3} = \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}, \quad e^{i\pi/6} = \tfrac{\sqrt{3}}{2} + i\tfrac{1}{2}$$

From those three equations it is easy to derive similar *Cartesian* expressions for

$$e^{-i\pi/4}, \quad e^{i\pi 3/4}, \quad e^{-i\pi 3/4}, \quad e^{-i\pi/3}, \quad e^{i2\pi/3}, \quad e^{-i2\pi/3}, \quad e^{-i\pi/6}, \quad e^{i5\pi/6}, \quad e^{-i5\pi/6}$$

Make a sketch of the unit circle and mark the arguments $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}$, etc.

Elementary functions Complex elementary functions having the same name as real elementary functions (such as exp, sin, cos, tan, cot) agree with these when applied to a real variable; hence for $x \in \mathbb{R}$ the values of $e^x, \sin x, \cos x, \tan x, \cot x$ are as before. There is one exception: extraction of roots (§1.5). The expression $\sqrt[n]{z} = z^{1/n}$ refers to any complex number which raised to the n-th power equals z, unless the *principal branch* or some other *branch* is specified.

Domain (§1.6). A *domain* is an open connected subset of the plane. An *open disc* is a domain whose boundary is a *circle*.

Arctan (p. 14). In the textbook the expression $\tan^{-1}$ is used instead of Arctan, the principal branch of arctan, that is the inverse function to

$$\tan :]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\to \mathbb{R}$$

In Maple arctan refers to the principal branch.

3 Problem session

Exercise A Representation of complex numbers in Cartesian and polar form. Given the complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = 1 + i$ in Cartesian form.

1. Determine *the absolut values* $|z_j|$ and *the principal arguments* $\operatorname{Arg} z_j$ of z_j for $j = 1, 2$.
2. Express z_j and $1/z_j$ $(j = 1, 2)$ in polar form, and sketch their positions in the plane. Express $1/z_j$ $(j = 1, 2)$ in Cartesian form.
3. Let n and m denote positive integers. For which n and m is it true that $(\sqrt{3} - i)^m = (1 + i)^n$? (Hint: Use polar representation.)

Exercise B Point sets in the plane.

1. Find equations of the form $|z - z_0| = \sqrt{2}$ for the circles passing through the two points i and $-i$ (begin by making a sketch).
2. Sketch the point set

$$\left\{ z \in \mathbb{C} \mid |z - 1| \leq \sqrt{2} \ \wedge \ |z + 1| \geq \sqrt{2} \right\}$$

3. Sketch the point set determined by the equation $|z - 1| = |z - i|$. Do the same for the inequality $|z - 1| > |z - i|$.

Exercise C Triangle inequality and opposite triangle inequality.

For any two complex numbers z_1, z_2 we have *the triangle inequality*:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and *the opposite triangle inequalities*:

$$|z_1 + z_2| \geq |z_1| - |z_2|$$

$$|z_1 + z_2| \geq |z_2| - |z_1|$$

Use these inequalities to find real numbers m and M so that

$$m \leq \left|z^3 - z\right| \leq M \quad \text{for all } |z| = 2$$

Are the inequalities best possible, i.e. do numbers z_1 and z_2 exist such that $\left|z_1^3 - z_1\right| = m$ and such that $\left|z_2^3 - z_2\right| = M$?
Also solve the problem when $|z| = 2$ is replaced by $|z| = \frac{1}{2}$.

Exercise D The complex exponential.

1. For $z = x + iy$ express the values of the following functions of z in terms of well-known real functions of x and y:
   $\operatorname{Re} e^z$, $\operatorname{Im} e^z$, $|e^z|$, $\arg e^z$ (all arguments), $e^{\bar{z}}$, $e^{|z|}$.
2. For which $w \in \mathbb{C}$ can the equation $e^z = w$ be solved?
3. Find all solutions to the equation $e^z = ie$, and all solutions to the equation $e^z = i(1 - e)$.

Exercise E Related to Möbius transformations, i.e. expressions of the form $\frac{az+b}{cz+d}$.

Consider the curve parametrized by $z(t) = \frac{t-i}{t+i}$ where $t \in \mathbb{R}$.

1. Show that $|z| = 1$ for all t.
2. Determine the position of $z(-1), z(0)$ and $z(1)$.
3. How are $z(t)$ and $z(-t)$ situated relative to each other ?
4. Explain why $z(t)$ describes the unit circle $|z| = 1$ except for the point at $z = 1$ when t describes $\mathbb{R}$.
   Sections 7.3-7.4 are devoted to Möbius transformations.

Exercise F DO NOT use inequalties < and > between complex numbers.

The rules for addition and multiplication are exactly the same for complex and real numbers. The real numbers are partitioned into *positive* and *negative* numbers and 0 in such a way that if x is positive then $-x$ is negative, and vice versa. Moreover, the real numbers are *ordered*, we define

$$x_1 < x_2 \iff x_2 - x_1 > 0$$

The ordering relation satisfies

$$x_1, x_2 > 0 \implies x_1 \cdot x_2 > 0$$

$$x_1, x_2 < 0 \implies x_1 \cdot x_2 > 0$$

$$x_1 > 0, x_2 < 0 \implies x_1 \cdot x_2 < 0$$

Show that it is NOT possible to partition the complex numbers into positive and negative numbers plus 0 in such a way that the ordering relation satisfies the three inequality rules above.

4 Homework problems

These problems are all exercises from the textbook. Note that for odd-numbered problems a short answer is given at the end of the book, if it is not already provided in the formulation of the problem. On Wednesday September 10 short comments to the problems will be posted on the course homepage under the menu item "Solutions to homework problems". Maple solutions to the problems can be found on the homepage under the menu item "Notes".

Make a habit of sketching whenever possible.

1. § 1.2 Exercise 5. An equilateral triangle. Sketch the triangle.
2. § 1.2 Exercise 7. Expressions of point sets in the plane. The point sets consist of all z-values satisfying the given equation or inequality. Think geometrically and interpret $|z - z_0|$ as the distance of z from z_0. Sketch the different point sets in the plane.
   To answer **(e)**, **(f)**, and **(g)** you are supposed to know the geometrical characterization of a parabola, an ellipse, and a circle of proportion (in Danish: *forholdscirkel*). They are given here.
   (a) Hint to **(e)**. Geometric characterization of a parabola: the set of points whose distance to a given point (*the focal point*) is equal to its distance to a given line (*the directrix*).
   (b) Hint to **(f)**. Geometric characterization of an ellipse: the set of points whose sum of distances to two distinct given points (the focal points) is equal to a given constant (this constant is equal to the length of the *major axis*).
   (c) Hint to **(g)**. Geometric characterization of a circle of proportion: the set of points whose distance to one given point is proportional to its distance to another given point with a fixed proportion. The diameter of the circle can easily be determined.
3. § 1.2 Exercise 17. Complex conjugate roots of a real polynomial.
4. § 1.3 Exercise 7. Arguments of complex numbers and polar form. Sketch the different points in the plane.
   The answer at the end of the book uses cis (v) as an abbreviation for the expression $\cos v + i \sin v$. This abbreviation is used only in §1.3. In the rest of the book e^{iv} is used.
5. § 1.4 Exercise 1. From polar to Cartesian form. Sketch the different points in the plane.
6. § 1.4 Exercise 7. Periodicity of the exponential function.
7. § 1.4 Exercise 17. Parametrizations of circles or arcs of a circle. Sketch the different curves in the plane.