Complex Analysis 01141

Department of Mathematics

Week 2, 2008

1 Coverage next week

In **3rd week** we cover §§ 2.3 − 2.4 and 3.1. In §§2.3 − 2.4 we show that the class of analytic functions corresponds to the class of real-differentiable vector functions from $\mathbb{R}^2$ (or an open subset of $\mathbb{R}^2$) to $\mathbb{R}^2$ which satisfy *the Cauchy-Riemann equations*. In §3.1 we study the basic elementary functions: polynomials and rational functions.

2 Comments on the material for next week

Analytic functions § 2.3. By definition **a real function** f is called *analytic at a point* x_0, if f is infinitely often differentiable in an interval around x_0 **and** if in this interval $f(x)$ is equal to its Taylor expansion

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Note, that a complex function can be *differentiable* at a single point z_0 without being analytic at z_0, since "f analytic at z_0" means differentiable at all points in some disc centered at z_0. Only if a function is analytic at z_0, can we differentiate the function at z_0 again and again and obtain the many results listed above, leading to the agreement with the definition of analyticity of a real function.

Poles and singular parts of rational functions § 3.1. A rational function R is given as the fraction $\frac{P}{Q}$ of two polynomials P and Q. Assume P and Q have no common roots. A root ζ of *muliplicity* d of the denominator polynomial Q is called a *pole of order* d of the rational function R. In § 5.6 Definition 8 pp. 278 - 279 we define poles in general. Note that formula (21) p. 106 gives a formula to determine any of the coefficients $A_s^{(j)}$ in the partial fraction decomposition described in Theorem 2 p. 105. Of special importance later on is the coefficient A_{d-1} to $\frac{1}{z-\zeta}$ where ζ denotes any of the poles of R. This coefficient is called the *residue* of R at ζ. In § 6.1 Definition 1 p. 308 we define residues in general. Note that expression (19) in Theorem 2 p. 105 can be rewritten for any of the poles ζ of order d as

$$R(z) = \frac{A_0}{(z-\zeta)^d} + \frac{A_1}{(z-\zeta)^{d-1}} + \cdots + \frac{A_{d-1}}{(z-\zeta)} + S_\zeta(z)$$

where $S_\zeta(z)$ is the sum of the remaining terms in (19); $S_\zeta(z)$ is defined in all of $\mathbb{C}$ except the poles different from ζ. The part $R(z) - S_\zeta(z)$ is called the *singular part* of R around ζ.

Limits including the point at ∞ Any rational function R can be extended by continuity to a function

$$R : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

defined on the extended complex plane and with values in the extended complex plane, compare with the footnote on p. 55 and the last paragraph on p. 62. For any pole ζ of R we have

$$\lim_{z \to \zeta} R(z) = \infty, \quad \text{hence we define} \quad R(\zeta) = \infty$$

Moreover, we define

$$R(\infty) = \lim_{z \to 0} R\left(\tfrac{1}{z}\right)$$

3 Problem session

Exercise A The complex exponential, again.

1. From the definition of e^{iy} when $y \in \mathbb{R}$

$$e^{iy} = \cos y + i \sin y$$

   it follows that

$$e^{i\pi} = -1$$

   Determine all z that satisfy $e^z = -1$.

2. We know that $e^{\ln 2} = 2$, and that the only real solution to the equation $e^x = 2$ is $x = \ln 2$. Determine all complex solutions z to the equation $e^z = 2$.
3. Now the general case: Let $w_0 \neq 0$ and z_0 be given so that $e^{z_0} = w_0$. Determine all complex solutions z to the equation $e^z = w_0$ in terms of z_0.
4. Note that $|e^z| = e^x = e^{\operatorname{Re} z}$. Determine $\left|e^{z^3+iz}\right|$ as a function of $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Exercise B Complex roots.
Notation: Both $z^{1/n}$ and $\sqrt[n]{z}$ are used to denote all possible ζ-values such that $\zeta^n = z$. Determine in polar form $re^{i\theta}$ all values of

$$\textbf{(a)}\ (2i)^{1/2} = \sqrt{2i} \quad \textbf{(b)}\ (-1)^{1/3} = \sqrt[3]{-1}$$

$$\textbf{(c)}\ 16^{1/4} = \sqrt[4]{16} \quad \textbf{(d)}\ \left(\tfrac{2i}{1+i}\right)^{1/3} = \sqrt[3]{\tfrac{2i}{1+i}}$$

Sketch is each case the position of all roots; the arguments of the roots are important and should be drawn rather accurately, whereas the sizes of the absolute values are of minor importance in the sketch.

Exercise C The Riemann sphere and the extended complex plane.
Let Σ denote *the Riemann sphere* in 3-dimensional space, i.e. the points $(x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying

$$x_1^2 + x_2^2 + x_3^2 = 1$$

The points on $\Sigma \setminus \{N\}$ are in one-to-one correspondence with points in the complex plane $\mathbb{C}$ through *stereographic projection*, see (1) p. 46 and (4) p. 47, and the point N corresponds (by definition) to *the extended complex number* ∞. Often the Riemann sphere Σ is identified with *the extended complex plane* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Describe the stereographic projection on Σ of the following sets, by expressions in the coordinates x_1, x_2, x_3 and in geometrical terms:

1. the real axis including the point at ∞
2. the imaginary axis including the point at ∞
3. the unit disk $|z| < 1$
4. the right half-plane Re $z > 0$

5. the disk $|z| < \frac{1}{2}$
6. the annulus $1 < |z| < 2$
7. the set $|z| > 3$ including the point at ∞

Exercise D Point sets in the plane.
Sketch the following point sets in the plane; use dots for parts of the boundary which are not parts of the set.

1. $|z - 2 + i| \leq 1$
2. $|2z + 3| > 4$
3. $|\text{Im}\, z| > 2$
4. $1 < |z - i| \leq 2$

Determine in each case if the point set is a *domain*. Determine if the set has one or more of the following properties: *open, closed, connected, bounded.*

Exercise E The principal argument Arg.
The principal argument $\text{Arg}(z)$ is defined for all $z \neq 0$ as the argument of z in the interval $]-\pi, \pi\,]$. Explain why Arg is discontinuous at every point on the negative real axis and continuous elsewhere.

Exercise F Multiplication with complex numbers in matrix form.
The function $f : \mathbb{C} \to \mathbb{C}$ given by $w = f(z) = cz$ where c is a fixed complex number, can also be interpreted as a vector function from $\mathbb{R}^2$ into $\mathbb{R}^2$. Explain why this function can expressed in matrix form as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $c = a + ib,\ z = x + iy,\ w = u + iv.$
Assume $c \neq 0$, and write the inverse function $z = f^{-1}(w) = c^{-1}w$ in matrix form.

Exercise G Limits in $\mathbb{C} \cup \{\infty\}$.
Find each of the limits

$$\lim_{z\to 2i} \frac{z}{(z-2i)^2}\ ; \qquad \lim_{z\to\infty} \frac{3z+4}{z-2i}\ ; \qquad \lim_{z\to 2i} \frac{3z+4}{z-2i}\ ; \qquad \lim_{z\to 2i} \frac{z^2+4}{z-2i}$$

4 Homework problems

Solutions to homework problems will be posted under the menu item “Solutions to homework problems” on the homepage before we meet next time.
Make sketches whenever possible!

1. § 1.3 **Exercise 12. The principal argument.**
Sketch the positions of the points. This will allow you to write down the principal argument immediately. If you are using Arctan, then be aware of the comments on p. 17^4 - 14^{13}.

2. § 1.4 **Exercise 2 and 4. Complex numbers in Cartesian and polar forms,** i.e. in the form $z = x + iy$ respectively $z = re^{i\theta}$.

3. § 1.5 **Exercise 4. Application of De Moivre’s formula.**

4. § 1.6 **Exercise 2 to 6. Point sets in the complex plane.**

5. § 2.1 **Exercise 6. Joukowski mapping.**
   (Hint to **(b)** and **(c)**: Express z in polar form $z = re^{i\theta}$ and its image $w = J(z)$ in Cartesian form $w = u + iv$.)
   For $r > 0$ let C_r denote the circle $|z| = r$. Note that $J(C_r) = J(C_{1/r})$.
   This mapping will reappear in later homework problems. It is also related to Example 1 pp. 437 - 439.

6. § 2.1 **Exercise 17. Inversion:** $z \mapsto \frac{1}{z}$**.**
   Let Σ denote the Riemann sphere and let $S : \mathbb{C} \cup \{\infty\} \to \Sigma$ be the stereographic projection. From Example 2 pp. 47 - 48 and the definition of the extended complex number ∞ on p. 48 it follows that all circles and all lines including ∞ are mapped by S onto circles in Σ, and vice versa. Obtain the result stated in the exercise by geometrical reasoning without further calculations.