Complex Analysis 01141

Department of Mathematics

Week 3, 2008

1 Coverage next week

In the **4th week** we cover §§ 3.2−3.3 and 3.5. We define the elementary trigonometric functions $\sin z, \cos z, \tan z, \cot z$ as well as the multiple-valued functions: the complex logarithm $\log z$, the complex powers z^α for any fixed complex number α, and the inverse trigonometric (multiple-valued) functions $\sin^{-1} z = \arcsin z, \cos^{-1} z = \arccos z, \tan^{-1} z = \arctan z$. We define *the principal branch* of the logarithm denoted Log; *the principal branches* of powers (no special notation), and the principal branch of $\sin^{-1}$ denoted Sin^{-1} or Arcsin. Note that principal branches are all defined in term of Log.

2 Comments on the material for next week

Trigonometric functions For the real trigonometric functions $\cos t, \sin t, \tan t, \cot t$ with $t \in \mathbb{R}$ it is natural to consider t as an angle and interpret the values of the functions geometrically. For the corresponding complex functions $\cos z, \sin z, \tan z, \cot z$ with $z \in \mathbb{C}$ it makes no sense to interpret z as an angle.
For any complex value of w we can solve the equation $\cos z = w$. The same holds for the equation $\sin z = w$. For any complex value of $w \neq \pm i$ the equations $\tan z = w$ and $\cot z = w$ can also be solved. All of these equations have for given w infinitely many z-solutions. As a consequence $\sin^{-1}, \cos^{-1}, \tan^{-1}, \cot^{-1}$ denote *multiple-valued functions*, while Sin^{-1}, Cos^{-1}, Tan^{-1}, Cot^{-1} denote *the principal branches* and are the complex extensions of the well-known real functions Arcsin, Arccos, Arctan, Arccot.

Domain of definition and domain of analyticity for branches of the logarithm *The principal branch* of the logarithm, Log, and any *branch*, $\mathcal{L}_\tau$, of the logarithm

$$\mathcal{L}_\tau(z) = \ln|z| + i \arg_\tau(z) \text{ with } \arg_\tau(z) \in \,]\tau, \tau + 2\pi],$$

is <u>defined</u> in the punctured plane $\mathbb{C} \setminus \{0\}$, but only <u>analytic</u> in the slit planes

$$D^* = \mathbb{C} \setminus \,]-\infty, 0] \quad \text{and} \quad D^*_\tau = \mathbb{C} \setminus \{ \, z = re^{i\tau} \mid r \geq 0 \, \}, \text{ respectively}$$

For any function defined using a branche of the logarithm (such as the principal branch of z^i, i.e. $e^{i \text{ Log } z}$) there is a similar difference between domain of definition and domain of analyticity.
The names arg and log are generally used to denote the *multiple-valued functions*, while Arg and Log denote the principal branches of arg and log, respectively, and $\arg_\tau$ and $\mathcal{L}_\tau$ other branches of arg and log, as specified above. The principal branch $\mathrm{Log} = \mathcal{L}_{-\pi}$ is the complex extension to D^* of the real logarithm ln defined on $\mathbb{R}_+$. Therefore the restriction of Log to $\mathbb{R}_+$ is equal to ln. The textbook uses the name Log for the real logarithm and not ln. In problem sheets and weekly worksheets ln is used to denote the real logarithmic function.

Unique extensions of the well known real functions to complex analytic functions

The well-known real elementary functions that we discuss (the exponential, the logarithm, the trigonometric functions and the power functions) are all extended to complex analytic functions in the entire complex plane or part of it. A natural question to ask is whether this could have been done in a different way than the definitions presented. The surprising answer is: NO! This follows from a result (often referred to as the identity theorem) outside the course syllabus, namely Corollary 5 p. 297:

The identity theorem. If f and g are analytic functions in a domain D and $f(z_n) = g(z_n)$ for an infinite sequence of distinct points $\{z_n\}$ converging to a point z_0 in D, then $f \equiv g$ throughout D.

Notice that the assumptions in the theorem/corollary are satisfied if f and g are two analytic functions that agree for all x in $\mathbb{R}$ or in some open interval of $\mathbb{R}$.

3 Problem session

Exercise A Analytic functions.

From Example 3 p. 68, we know that for any positive integer $n \in \mathbb{N}$ the function $f(z) = z^n$ is analytic in the plane with derivative $f'(z) = nz^{n-1}$. Use the rules **(5)** – **(9)** in Theorem 3 p. 69 to determine the derivatives of the following functions (do not spend time on reducing the expressions as much as possible, just obtain a correct expression):

1. $f(z) = 6z^3 + iz + 10$
2. $f(z) = (z^2 - 3i)^2$
3. $f(z) = \frac{6z^3+iz+10}{(z^2-3i)^2}$
4. $f(z) = z^{-6}$

Determine in each case the set of points in which the function is defined and the set where it is analytic. (In the cases above it turns out that the two sets coincide, but this not always the case.)

Exercise B Cauchy-Riemann equations.

Find for each of the following functions f : the real part $u = \text{Re } f$ and the imaginary part $v = \text{Im } f$, and the set of points at which the function is differentiable. Find the derivative $f'(z)$ at these points. Determine where the function is analytic.

1. $f(z) = \cos x \cosh y - i \sin x \sinh y$
2. $f(z) = e^x e^{-iy}$
3. $f(z) = x^3 + iy^3$

Exercise C Real analytic functions are constant.

Suppose $f : D \to \mathbb{C}$ is analytic in the domain D, and that $f(z)$ only takes real values, i.e.

$$\text{Im } f(z) = 0 \quad \text{for all } z \in D.$$

Show, that f is constant. (Hint: Use the Cauchy-Riemann equations.)

Exercise D Limits in $\mathbb{C} \cup \{\infty\}$ and continuity of simple rational functions.

Given a rational function

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc \neq 0 \ .$$

Such a function is called a *Möbius transformation* and will be studied further in §§7.3−7.4.

1. Assume $c \neq 0$. Determine $\lim_{z\to\infty} f(z)$ and $\lim_{z\to -d/c} f(z)$. Explain how f can be extended by continuity to a map

$$f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

by defining $f(\infty) = \lim_{z\to\infty} f(z)$ and $f(-\frac{d}{c}) = \lim_{z\to -d/c} f(z)$.

2. Assume $c = 0$. Determine $\lim_{z\to\infty} f(z)$, and explain how f can be extended by continuity to a map

$$f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \ .$$

<u>Comment:</u> When studying Möbius transformations in §§7.3 − 7.4. we shall always consider the mappings extended as above to $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.

Exercise E Expansions centered at $z_0 = i$.

Consider the function given by

$$f(z) = \frac{z+i}{(z-i)^2} + z^3 - 3z - 4.$$

f is the sum of a rational function R and a polynomial P; it is defined and analytic in $\mathbb{C} \setminus \{i\}$. Explain why the function can be expressed as an expansion in powers of $z - i$ of the form

$$f(z) = \sum_{k=-2}^{3} a_k (z-i)^k$$

where $k = -2$ is the order of the pole of R at $z_0 = i$ and 3 is the degree of P. Determine the coefficients a_0, a_1, a_2, a_3 by the formula for P given in (12) p. 103 and the coefficients a_{-2}, a_{-1} by the formula for R given in (21) p. 106. (Remember that $0! = 1$. By $\frac{d^0}{dz^0}$ is meant no differentiation.)

Exercise F Triangle inequality and opposite triangle inequality, again.

1. Write the polynomial

$$z^4 - 4iz^2 - 3$$

as a product of two 2nd degree polynomials.

2. Show (by using the triangle inequality and its opposite) that the following inequalities hold for all $|z| = 2$:

$$3 \leq \left|z^4 - 4iz^2 - 3\right| \leq 35$$

$$7 \leq \left|z^3 - 1\right| \leq 9.$$

3. Use these inequalities to conclude that

$$\tfrac{1}{5} \leq \frac{\left|z^3 - 1\right|}{\left|z^4 - 4iz^2 - 3\right|} \leq 3 \quad \text{for all } |z| = 2.$$

Remark: Often during the course we need to estimate absolute values of a function of the form

$$f(z) = \frac{g(z)}{h(z)}.$$

4 Homework problems

On Wednesday, September 24, short comments to the problems will be posted under "Solutions to homework problems" on the course homepage. Also see the Maple solutions.

1. § 2.1 **Exercise 3. Range of functions.**
   Sketch the given sets in the z-plane and the corresponding range of functions in the w-plane.

2. § 2.2 **Exercise 11. Limits.**
   The answer to (d) is correct only if we change the problem. Change (d) to read:

$$\lim_{z \to i} \frac{z^2 + 1}{z^4 - 1}$$

3. § 2.2 **Exercise 17. Example of no limit.**

4. § 2.3 **Exercise 9. Analyticity.**
   You are asked to determine the points at which the function is not analytic. The set of points at which the function *is* analytic is called the *domain of analyticity*.

5. § 2.4 **Exercise 1. Examples of nowhere differentiable functions.**

6. § 3.1 **Exercise 11. Poles of rational functions and their multiplicity.**