Complex Analysis 01141

Department of Mathematics

Week 5, 2008

1 Coverage next week

In the **6th week** we cover §§ 7.3 - 7.4 (to the end of Example 2 p. 400) which focus on *Möbius transformations*. The Möbius transformations are conformal bijective mappings of the extended complex plane onto itself.

2 Comments on the material for next week

Bijective mappings of the extended complex plane $C \cup \{\infty\}$ onto itself. §§ 7.3 − 7.4 deal with *Möbius transformations*, i.e. mappings of the form $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$. For $c \neq 0$ the transformation f is a conformal bijective mapping of $\mathbb{C} \setminus \{-\frac{d}{c}\}$ onto $\mathbb{C} \setminus \{-\frac{a}{c}\}$. By defining $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ the transformation f is extended by continuity to a bijective mapping of *the extended complex plane* $\mathbb{C} \cup \{\infty\}$ onto itself. For $c = 0$ the transformation f is of the form $f(z) = az + b$ and a conformal bijective mapping of $\mathbb{C}$ onto $\mathbb{C}$. By defining $f(\infty) = \infty$ this mapping also becomes a continuous bijective mapping of the extended complex plane onto itself. These extentions were discussed in Excersize E, 3rd problem sheet.
We always consider Möbius transformations as defined on the extended complex plane.

Generalized circles A *generalized circle* is defined as a circle or a straight line including the point at ∞ (see p. 388_{13-17}). Many formulations are simplified when using the notion of "generalized circles". For instance (iii) in Theorem 5 p. 390 reads better and more precisely as:
"*f is mapping a generalized circle onto a generalized circle*".
Since three different points determine a generalized circle, it is easy to determine the image of a generalized circle under a Möbius transformation: The generalized circle through z_1, z_2, z_3 is mapped onto the generalized circle through $w_1 = f(z_1), w_2 = f(z_2), w_3 = f(z_3)$. Any generalized circle in the extended complex plane corresponds under stereographic projection to a circle on the Riemann sphere (compare with figure 1.23 p. 47) and vice versa.

Conformality Any Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ corresponds through stereographic projection to a mapping $F : \Sigma \to \Sigma$ from the Riemann sphere onto the Riemann sphere, mapping circles onto circles. The mapping F is conformal (angle preserving) everywhere. Hence we say that f is conformal everywhere. This can also be justified by the conformality of compositions with the inversion $I : z \to \frac{1}{z}$. If $c \neq 0$ then $I \circ f$ is conformal at $-\frac{d}{c}$ and $f \circ I$ is conformal at 0, corresponding to f being conformal at $-\frac{d}{c}$ respectively ∞. If $c = 0$ then $I \circ f \circ I$ is conformal at 0, corresponding to f being conformal at ∞.

Determining the image of a domain from the image of its boundary. Note that f is mapping an arbitrary domain $A \subset \mathbb{C} \cup \{\infty\}$ conformally and bijectively onto its image $f(A)$. Knowing the image of the boundary of A there are two ways to determine $f(A)$, either by knowing the image of just one point z_0 in A and using that $f(A)$ is connected, or by considering an orientation of ∂A which induces an orientation of $f(\partial A)$ (compare with figures 7.24 and 7.25 on pp. 398 - 399).

Use of Appendix B Appendix B contains a table of *conformal bijective* mappings from certain domains in a z-plane (marked in grey or hatched) onto other domains in a w-plane (analogously marked in grey or hatched). The boundaries of the domains are highlighted (but are not parts of the domains). The image $f(a)$ of a point $z = a$ is marked as the point $w = f(a) = A$. Similarly for $f(b) = B, f(c) = C, \ldots$

Inverses and compositions of conformal bijective mappings are again conformal and bijective. Therefore the examples can be combined in many different ways. As an illustration the third example on p. A-23 is a composition of the first example of p. A-20 with the inverse of the last example of p. A-22.

3 Problem session

Exercise A Harmonic functions and harmonic conjugates.

Consider the real functions $u_j : \mathbb{R}^2 \to \mathbb{R},\ j = 1, 2$, given by

$$u_1(x, y) = 2x(1 - y) \quad \text{and} \quad u_2(x, y) = x^3 - 3xy^2$$

1. Show by applying Definition 6 p. 79 that these functions are harmonic.
2. Determine for each of the functions u_j all harmonic conjugates v_j (i.e. functions v_j making $f_j = u_j + iv_j$ an analytic function). Write f_j as a function of z where $z = x + iy$.

Exercise B $\arg_\tau(z)$ **and** $\ln|z|$ **as harmonic functions.**

1. Consider any branch of the logarithm of the form

   $$\mathcal{L}_\tau(z) = \ln|z| + i \arg_\tau(z), \quad \text{where } \arg_\tau(z) \in]\tau, \tau + 2\pi].$$

   Explain why $v(z) = \arg_\tau(z)$ is a harmonic function in the slit plane $D^*_\tau = \mathbb{C} \setminus \{re^{i\tau} \mid r \geq 0\}$, but not in $\mathbb{C} \setminus \{0\}$.
   This is an extension of Corollary 1 p. 121.
2. Let D be a domain given as the union of two domains $D = D_1 \cup D_2$. If $u : D \to \mathbb{R}$ is known to be harmonic in D_1 and in D_2, then conclude that u is harmonic in D.
   Use this fact to explain (without calculations) why $u(z) = \ln|z|$ is a harmonic function in $\mathbb{C} \setminus \{0\}$.
   This is the content of Corollary 2 p. 121.

Exercise C A Dirichlet problem.

Solve Exercise 2 in § 3.4.

Exercise D Conformal bijective mappings.

Appendix B pp. A-19 − A-25 is a table of conformal mappings $f : D \to \mathbb{C}$ mapping a domain D (shown in grey or hatched) in a z-plane *conformally and bijectively* onto a domain $E = f(D)$ (likewise shown in grey or hatched) in a w-plane. In all cases the mapping is actually defined in a larger set, in particular including the boundary of D. The boundaries of D and E are highlighted (but are not parts of the domains). Moreover, the image $f(a)$ of a point $z = a$ is marked as the point $w = f(a) = A$. Similarly for $f(b) = B, f(c) = C, \ldots$

1. Consider the first example on p. A-22:

   $$w = f(z) = z^2$$

and the domain, the half strip

$$D = \{ \ z = x + iy \mid 0 < x < 1 \wedge y > 0 \ \}$$

The domain D is bounded by the three curves

$$\gamma_1 \ : \ z = iy, \ y \geq 0 \ ; \quad \gamma_2 \ : \ z = x, \ 0 \leq x \leq 1 \ ; \quad \gamma_3 \ : \ z = 1 + iy, \ y \geq 0$$

Determine the images $f(\gamma_j)$, $j = 1, 2, 3$, of the curves in the w-plane (compare with the figure). Show that f is conformal in D and mapping D injectively, hence bijectively onto $E = f(D)$. Identify the inverse function $f^{-1} : E \to D$ as a well-known function.

2. Consider the last example on p. A-22:

$$w = f(z) = e^z$$

and the domain, the strip

$$D = \{ \ z \mid 0 < \text{Im } z < \pi \ \}.$$

Show that f is conformal in D and mapping D injectively, hence bijectively onto the upper half plane $E = f(D)$. Identify the inverse function $f^{-1} : E \to D$ as a well-known function.

Exercise E The principal branch Arcsin.

On p. 134 in Example 3 all solutions to the equation $\sin w = z$ are determined. The infinitely many solutions are expressed by the multiple valued function

$$\sin^{-1} z := \arcsin z := -i \log[iz + (1 - z^2)^{1/2}]$$

The *principal branch* is defined on p. 135, in Example 4, as

$$\text{Sin}^{-1} \ z := \text{Arcsin } z := -i\text{Log}[iz + e^{(1/2)\text{Log}(1-z^2)}]$$

One can show that this function is mapping the double slit plane

$$D = \mathbb{C} \setminus \ (] - \infty, -1] \cup [+1, +\infty[)$$

conformally and bijectively onto the strip $S = \{ \ w = u + iv \mid |u| < \frac{\pi}{2} \ \}$. The principal branch is the complex extension of the real function

$$\text{Arcsin } :] - 1, 1[\ \to \ ] - \tfrac{\pi}{2}, \tfrac{\pi}{2} \ [$$

Instead of discussing Arcsin directly we consider the sine function in the first example on p. A-24.

1. Show that the vertical half-lines which bound the half strip S_+ given as

$$S_+ = \{ \ z = x + iy \mid |x| < \tfrac{\pi}{2} \ \wedge y > 0 \ \}$$

are mapped by the sine function as shown on p. A-24. Show that the sine function is conformal in S_+ . The sine function is mapping S_+ injectively, hence bijectively onto its image $\sin(S_+)$. (You are not asked to prove the injectivity.)
Show that $\sin(S_+)$ is equal to the upper half-plane. (Hint: Determine $\sin i$.)

2. Show that $\sin \overline{z} = \overline{\sin z}$.

3. Extend the figure on p. A-24 to include the whole strip S.

4 Homework problems

On Wednesday, October 8, a solution to the homework problems will be posted under the menu item "Solutions to homework problems" on the course homepage.

1. § 2.5 **Exercise 17. Two Dirichlet problems.**

2. § 3.2 **Exercise 5. Using the definitions of the exponential, trigonometric and hyperbolic functions.**

3. § 3.3 **Exercise 1. Using the definition of the multiple-valued logarithm and the principal branch of the logarithm.**

4. § 3.3 **Exercise 3. Be careful when using principal branches.**
   The well-known formula

   $$\ln(x_1 x_2) = \ln x_1 + \ln x_2 \quad \text{for all } x_1, x_2 > 0$$

   generalizes to the multiple-valued formula

   $$\log(z_1 z_2) = \log z_1 + \log z_2 \quad \text{for all } z_1, z_2 \neq 0.$$

   (There is a misprint in the book: 'y' after 'then' should be erased.)

5. § 3.5 **Exercise 7. Differentiation of powers.**

6. § 3.5 **Exercise 15 (a) and (b). Determination of principal branches which are analytic in given domains.**