Complex Analysis 01141

Department of Mathematics

Week 6, 2008

1 Coverage next week

In the **7th week** we work with complex integration §§ 4.1 − 4.3.

2 Comments on the material for next week

The Equivalence Theorem, Cauchy's Integral Theorem The Equivalence Theorem, Theorem 7 p. 176 in § 4.3, says that three statements (i), (ii), and (iii) are equivalent, it says nothing about their being actually true for a concrete function. But if we know that one of the statements is true for a function f, then the two other statements are true too. Example: $f(z) = \frac{1}{z^2}$ is analytic in $D = \mathbb{C} \setminus \{0\}$, has an antiderivative $F(z) = -\frac{1}{z}$ in D, hence (i) is true, and then also (ii) and (iii).
Cauchy's Integral Theorem, Theorem 9 p. 187 in § 4.4a, states that every function which is analytic in a simply connected domain satisfies (ii) in Theorem 7. Theorem 10 p. 187 then draws the conclusion from Theorem 7, that in a simply connected domain an analytic function has all three properties.
The condition "a simply connected domain" in Theorem 9 and 10 is called a *sufficient condition*, namely sufficient to draw the conclusion. The example above shows that it is not *necessary*: the domain $D = \mathbb{C} \setminus \{0\}$ is not simply connected, but $f(z) = \frac{1}{z^2}$ has an antiderivative in D.

Harmonic conjugates in simply connected domains Let D be a simply connected domain and $\phi : D \to \mathbb{R}$ be a harmonic function in D. Then

$$f(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = u + iv$$

is analytic in D since the first partial derivatives of u and v exist in D, are continuous and satisfy the Cauchy-Riemann equations. It follows from Theorem 10 that f has an antiderivative $F : D \to \mathbb{C}$. Any antiderivative is of the form $F + c$ where $c \in \mathbb{C}$ is a constant, including those where the real part is equal to the given harmonic function ϕ. It follows that in a simply connected domain a harmonic function ϕ always has harmonic conjugates. In a domain that is not simply connected we may not be able to find a harmonic conjugate, however this is always possible in any simply connected subdomain. For instance, the harmonic function $\ln|z|$ which is harmonic in $\mathbb{C} \setminus \{0\}$ has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$. Any branch $\arg_\tau$ of the argument is a harmonic conjugate in the slit plane $D^*_\tau = \mathbb{C} \setminus \{z = re^{i\tau} | r \geq 0\}$, a simply connected subdomain of $\mathbb{C} \setminus \{0\}$.

3 Problem session

Exercise A A Möbius transformation and generalized circles.
Given the Möbius transformation

$$f(z) = \frac{iz+1}{z+i}$$

Consider f as a mapping of the extended complex plane onto itself.

1. Determine the images under f of the 6 points $0,\ \pm 1,\ \pm i,\ \infty$, and sketch the result as $f(a) = A,\ f(b) = B, \ldots$, like in Appendix B.
2. Let $C_1 = \{\ z \in \mathbb{C} \mid |z| = 1\ \}$ denote the unit circle. Moreover, let $R = \mathbb{R} \cup \{\infty\}$ and $I = i\mathbb{R} \cup \{\infty\}$ denote the two generalized circles, obtained by adding the point ∞ to the real axis respectively the imaginary axis. Determine the generalized circles $f(C_1),\ f(R)$ and $f(I)$.
3. Let $\mathbb{D} = \{\ z \in \mathbb{C} \mid |z| < 1\ \}$ denote the unit disc and $\mathbb{H} = \{\ z \in \mathbb{C} \mid \operatorname{Im} z > 0\ \}$ the upper half plane. Determine the domains $f(\mathbb{D})$ and $f(\mathbb{H})$. (Hint: Use the method described in Example 3 pp. 390 - 391 or the one illustrated in figure 7.25 p. 399.)

Exercise B Solving a Dirichlet problem.
Let D denote the domain between the circles

$$C_1 : \quad |z-1| = 1 \quad \text{and} \quad C_2 : \quad |z-2| = 2.$$

1. Sketch D and show that D under the Möbius transformation

   $$w = f(z) = \tfrac{4-z}{z}$$

   is mapped onto the strip $S = \{\ w = u + iv \mid 0 < u < 1\ \}$.
   We wish to solve the following Dirichlet problem: Determine a function $\phi : \overline{D} \to \mathbb{R}$, harmonic in D and satisfying the boundary conditions given by

   $$\phi(x,y) = \begin{cases} k & \text{for } z \in C_1 \setminus \{0\} \\ 0 & \text{for } z \in C_2 \setminus \{0\} \end{cases}$$

   where k is a positive real constant. Instead of solving the given Dirichlet problem directly, we shall determine a suitable harmonic function $\psi : S \to \mathbb{R}$ so that $\psi \circ f$ solves the original Dirichlet problem.
2. Determine the boundary conditions of ψ on ∂S in order to satisfy the boundary conditions of ϕ on ∂D if $\phi = \psi \circ f$. We say that the new boundary conditions are induced by f.
3. Solve the new Dirichlet problem: Determine a function $\psi : \overline{S} \to \mathbb{R}$, harmonic in S and satisfying the boundary conditions on ∂S induced by f. Then state the solution ϕ to the original Dirichlet problem.

Exercise C The Riemann Mapping Theorem (pp. 380 - 381).
Let $\mathbb{H} = \{\ z \in \mathbb{C} \mid \operatorname{Im}\, z > 0\ \}$ denote the upper half plane. Consider the same Möbius transformation

$$f(z) = \frac{iz+1}{z+i}$$

as in **Exercise A** above.

1. Determine $f'(i)$, and explain why f is the only conformal bijective mapping of the upper half plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$ which maps $z = i$ onto $w = 0$ and the direction of the positive real axis through $z = i$ onto the direction of the positive real axis through $w = 0$.

2. Let $g : \mathbb{D} \to \mathbb{D}$ denote an arbitrary conformal bijective mapping. Explain why $g \circ f : \mathbb{H} \to \mathbb{D}$ is a conformal bijective mapping, and express $(g \circ f)^{-1}$ in terms of g^{-1} and f^{-1}.
   Let $h : \mathbb{H} \to \mathbb{D}$ be an arbitrary conformal bijective mapping of the upper half plane onto the unit disc. Explain why h can always be obtained as a composition $h = g \circ f$ with f as above and g a suitable conformal bijective mapping of the unit disc onto itself.

Conclusion: All conformal bijective mappings of the upper half plane onto the unit disc are of the form $h = g \circ f$ with f as above and g an arbitrary conformal bijective mapping of the unit disc onto itself.

Exercise D Composition of conformal bijective mappings.
Let θ, φ be given as two angles in $]0, \pi[$ and let $W_{\theta,\varphi}$ denote the wedge domain given as

$$W_{\theta,\varphi} = \{ \ z = re^{it} \mid r > 0, \varphi < t < \theta + \varphi \ \}.$$

Sketch the domain $W_{\theta,\varphi}$. Determine a conformal bijective mapping $f_{\theta,\varphi}$ mapping $W_{\theta,\varphi}$ onto the unit disc $\mathbb{D}$. (Hint: Make for instance a composition of conformal bijective mappings $W_{\theta,\varphi} \to W_{\theta,0} \to \mathbb{H} \to \mathbb{D}$ where $\mathbb{H}$ denotes the upper half plane.)

Exercise E Conformal bijective mappings of the unit disc $\mathbb{D}$ onto itself.
It can be shown that any conformal bijective mapping of the unit disc $\mathbb{D}$ onto itself is a Möbius transformation of the form

$$f(z) = e^{i\theta} \frac{z - \alpha}{\overline{\alpha} z - 1} \qquad \text{where } |\alpha| < 1.$$

1. Find all conformal bijective mappings $f : \mathbb{D} \to \mathbb{D}$ which map $z = \frac{i}{2}$ onto $w = 0$.
2. Determine a conformal bijective mapping $g : \mathbb{D} \to \mathbb{D}$ which maps $z = \frac{i}{2}$ onto $w = \frac{1+i}{2}$. Compare with figure 7.12 p. 381.

4 Homework problems

Solutions will be posted on the homepage no later than October 22.

1. § 7.3 **Exercise 3. Möbius transformations and generalized circles.**
   Compare with the comments on the 5th weekly worksheet.
   Split the exercise into two parts:
   (a) First determine the image of the circle $|z - 2| = 1$ for each of the five Möbius transformations. In each case the result is a generalized circle. (Hint: Use that the image of three different points on the circle is enough to determine the result.)
   (b) Then determine the image of the open disc $|z - 2| < 1$ for each of the five Möbius transformations. In each case the result is a domain. (Hint: Use the method described in Example 3 pp. 390 - 391 or the one illustrated in figure 7.25 p. 399.)

2. **The second extension to Exercise 6 in** § 2.1.
   Two branches of the inverse to the Joukowski function.
   The exercise can be solved independently of the previous ones about the Joukowski function.
   Consider the double slit plane

$$A = \mathbb{C} \ \setminus \ (] - \infty, -1] \cup [+1, +\infty[)$$

   and the functions

$$f_+(z) = z + i\sqrt{1 - z^2} \quad \text{and} \quad f_-(z) = z - i\sqrt{1 - z^2}$$

   where $\sqrt{}$ is chosen as the principal branch of the square root, i.e. $\sqrt{} = e^{\frac{1}{2}\mathrm{Log}}$.

(a) Explain why A is the domain of analyticity of f_+ and f_- . Show that f_+ and f_- are conformal in A.

(b) Show that for $z \in A$ the function values $f_+(z)$ and $f_-(z)$ are the two different solutions to the equation $J(w) = z$, i.e. $\frac{1}{2}(w + \frac{1}{w}) = z$. It follows that $f_\pm$ are analytic branches of J^{-1}.

(c) Let $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ and $\mathbb{H}_- = \{z \in \mathbb{C} \mid \text{Im } z < 0\}$ denote the upper and lower half plane, respectively. It follows from **(2)** to **(4)** in the first extension of Exercise 6 in § 2.1 (see 4th weekly sheet) that $f_+(A) \cup f_-(A) = \mathbb{C} \setminus \mathbb{R}$.
Explain why $f_+(A) = \mathbb{H}_+$ and $f_-(A) = \mathbb{H}_-$.
Hint: Use that a continuous function (here: $f_\pm$) is mapping a connected set (here: A) onto a connected set. Note that it suffices to examine the position of a single value of $f_\pm(z)$, for instance $f_\pm(0)$, to conclude that $f_+(A) \subseteq \mathbb{H}_+$ and $f_-(A) \subseteq \mathbb{H}_-$. Finally conclude from $f_+(A) \cup f_-(A) = \mathbb{C} \setminus \mathbb{R}$ that $f_+(A) = \mathbb{H}_+$ and $f_-(A) = \mathbb{H}_-$.

Conclusion. We have shown that f_+ is the inverse function of $J : \mathbb{H}_+ \to A$, and that f_- is the inverse function of $J : \mathbb{H}_- \to A$. The figures corresponding to the 2nd and 3rd example on p. A-22 illustrate (with corrected scalings), that $J : \mathbb{H}_+ \to A$ is a conformal bijective mapping.

3. § 7.1 **Exercise 4. Solving a Dirichlet problem.**
Hint: Compare with the description of Dirichlet problems on the 4th weekly worksheet. The “must” in **(b)** follows from Theorem 28 p. 223.