Complex Analysis 01141

Department of Mathematics

Week 8, 2008

1 Coverage next week

In the **8th week** we continue working with complex integration, § 4.4a and § 4.5.

2 Comments on the material for next week

The main emphasis is on theoretical results In finding values of integrals we concentrate on applications of theoretical results without doing much calculation. The fundamental results in Example 2 p. 166 are exceptions and should be known by heart.

The most important theorems are

- the Fundamental Theorem of Integration, Theorem 6 p. 173,
- the Deformation Invariance Theorem, Theorem 8 p. 186,
- Cauchy's Integral Theorem, Theorem 9 p. 187,
- Theorem 10 p. 187,
- Cauchy's Integral Formulas, Theorem 14 p. 204 and Theorem 19 p. 211 and
- Theorem 16 p. 209.

The last theorem states the important result that any analytic function is infinitely often differentiable. This is in contrast to properties of real differentiable functions; for each integer $k \geq 1$ one can find examples of real functions which are k times differentiable, but not $(k+1)$ times.
Theorem 14 and 19 can be formulated as one theorem, *the Generalized Cauchy Integral Formula*:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

where n takes on all values among the positive integers and zero. (Recall that $0! = 1$ and $f^{(0)} = f$.)

A simple closed curve An integral along a simple closed curve Γ means the integral along Γ, traversed once in counterclockwise direction, unless stated otherwise.

3 Problem session

Exercise A A Möbius transformation and a Dirichlet problem.

Let $R = \mathbb{R} \cup \{\infty\}$ and $I = i\mathbb{R} \cup \{\infty\}$ denote the extended real and imaginary axes, respectively, and let C denote the circle $|z - 5i| = 4$. Let D denote the domain consisting of all points in the upper half-plane outside the circle C.

Consider the Möbius transformation

$$w = f(z) = \frac{2z - 6i}{z + 3i}$$

and the following images under f

$$f(\infty) = 2, \quad f(i) = -1, \quad f(9i) = 1, \quad f(0) = -2, \quad f(1) = 2\left(-\frac{4}{5} - i\frac{3}{5}\right)$$

1. Sketch the domain D.
2. Find by using these images and the fact that f is conformal, the images of the generalized circles R, I, and C and conclude that the image of the domain D is the annulus given by
$$f(D) = \{w \,|1 < |w| < 2\}$$
3. Using the result above, solve the Dirichlet problem $\phi : \overline{D} \to \mathbb{R}$ satisfying the boundary conditions $\phi(\mathbb{R}) = 1, \phi(C) = 0$. (Hint: Compare with Example 1 pp. 125-126.)

Exercise B Direct computation of integrals and application of the Fundamental Theorem of Integration, Theorem 6 p. 173.

Consider the two half-circles with center at 0 and radius $\sqrt{2}$ and with initial point $z_I = 1-i$ and terminal point $z_T = -1+i$. Let γ_1 be the one that is oriented counterclockwise and γ_2 the one that is oriented clockwise.

1. Compute for $j = 1, 2$, the integral $\int_{\gamma_j} \frac{dz}{z}$ by substituting a parametrization of γ_j as in formula (5) p. 165. (Compare with Example 1 pp. 164-165.)
2. Determine for $j = 1, 2$, an antiderivative $F_j(z)$ of the function $f(z) = \frac{1}{z}$ in a domain D_j containing γ_j. Then compute $\int_{\gamma_j} \frac{dz}{z}$ by using Theorem 6 p. 173. Compare with Example 2 pp. 174-175.

Exercise C Integrals of the form $\int_{|z-z_0|=r} \frac{dz}{(z-z_0)^n}$. Compare with Example 2 p. 166.

Let C_2 denote the circle $|z-1| = 2$ oriented counterclockwise. Compute the three integrals

$$\int_{C_2} (z-2)\,dz, \quad \int_{C_2} \frac{dz}{z-1}, \quad \int_{C_2} \frac{dz}{(z-1)^2}$$

If possible, determine in each case an antiderivative to the integrand in a domain which contains the circle C_2.

Exercise D Estimate of an integral.

Given

$$P(z) = z^3 - 1, \quad Q(z) = \left(z^2 - 3i\right)\left(z^2 - i\right), \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)}$$

1. By using the triangle inequality and the opposite triangle inequality show that
$$|P(z)| \le 9 \quad \text{and} \quad |Q(z)| \ge 3 \quad \text{for all} \quad |z| = 2$$
Conclude that $|f(z)| \le 3$ for all $|z| = 2$.

2. Apply Theorem 5 p. 170 to obtain an upper estimate of the absolute value of the integral $\int_{|z|=2} f(z)\, dz$.
   The exact value of the integral can be found directly by methods shown in §4.4a and more generally in §6.1.

Exercise E Estimate of limits of integrals along circles as the radius tends to infinity.
Consider the principal branch of the logarithm Log: $\mathbb{C} \setminus \{0\} \to \mathbb{C}$

$$\text{Log } z = \ln|z| + i\text{Arg } z \text{ with } -\pi < \text{ Arg } z \leq \pi$$

1. Show that for z on the circle $|z| = R$ with $R > 1$, we have the estimate

$$|\text{Log } z| \leq \ln R + \pi$$

2. By applying Theorem 5 p. 170 show that

$$\left| \int_{|z|=R} \frac{\text{Log } z}{z^2} dz \right| \to 0 \text{ as } R \to +\infty$$

   and hence

$$\int_{|z|=R} \frac{\text{Log } z}{z^2} dz \to 0 \text{ as } R \to +\infty$$

   (You may use the well-known result that $\ln x/x \to 0$ as $x \to +\infty$.)

4 Homework problems

On Thursday, October 30 short comments to the problems will be posted on the course homepage.

1. § 7.1 **Exercise 1. Two inherited harmonic functions.**
2. § 7.2 **Exercise 11. Images of domains under the exponential function.**
   The logarithmic spiral given in polar coordinates in the answer to **(b)** p. A-43 is in complex notation given as $z = e^{t(1+i)}, t \in \mathbb{R}$.
   The second answer to **(f)** p. A-43 should be the punctured disc, i.e. $\{w : |w| < 1\} \setminus \{0\}$.
3. § 7.3 **Exercise 7. Möbius transformations determined by the images of three points.**
4. § 7.3 **Exercise 9. Image under Möbius transformation.**
   Sketch the sector in the z-plane and its image in the w-plane. The answer on p. A-44 is to be understood in $\mathbb{C} \cup \{\infty\}$.
5. § 4.3 **Exercise 5. An example of an analytic function which in a given domain has no antiderivative.**
6. § 4.3 **Exercise 7. Use of the Fundamental Theorem of Integration, Theorem 6, p. 173.**