Complex Analysis 01141

Department of Mathematics

Week 9, 2008

1 Coverage next week

In the **10th week** we focus on §§ 5.1 – 5.3 which cover series in general, and power series and Taylor series in particular.

2 Comments on the material for next week

Tests of convergence The *comparison test* and the *ratio test* are formulated in their complex versions in Theorem 1 p. 236 and Theorem 2 p. 237, respectively. You ought to review other tests of convergence known from real analysis: the *root test*, the *asymptotic equivalence test* and the *integral test*. Note that in all cases we use absolute values of complex numbers in the tests.

Main theorems Theorem 3 p. 243 implies that any analytic function $f : D \to \mathbb{C}$ is equal to its Taylor expansion around z_0 in the largest open disc in D centered at z_0. Conversely Theorems 10 and 11 p. 256 imply that every power series

$$\sum_{j=0}^{+\infty} a_j(z - z_0)^j \quad \text{with radius of convergence } R > 0$$

determines an analytic function f in the disc $|z - z_0| < R$. The power series equals the Taylor series of f. It follows that the radius of convergence R of the Taylor series of $f : D \to \mathbb{C}$ around z_0 is greater than or equal to the radius in the largest disc in D centred at z_0. If two power series are equal

$$\sum_{j=0}^{+\infty} a_j(z - z_0)^j = \sum_{j=0}^{+\infty} b_j(z - z_0)^j$$

in an open disc $|z - z_0| < r$ where $r > 0$, then $a_j = b_j$ for all $j = 0, 1, \ldots$ since the coefficients are determined as

$$a_j = \frac{f^{(j)}(z_0)}{j!}$$

where f is the analytic function determined by the power series in the disc.

A Taylor series around $z_0 = 0$ is also called a Maclaurin series.

Radius of convergence Only power series can be assigned a radius of convergence.

Interchanging summation and integration Suppose the partial sums $S_n(z)$ of an infinite series $\sum_{j=0}^{+\infty} g_j(z)$ converge uniformly to the function $S(z)$ on a set T. Let Γ be a contour in T. Since the integral of a finite sum is the sum of integrals we have

$$\int_\Gamma S_n(z)\,dz = \int_\Gamma \left(\sum_{j=0}^{n} g_j(z)\right) dz = \sum_{j=0}^{n} \left(\int_\Gamma g_j(z)\,dz\right)$$

Using the uniform convergence it follows from Theorem 8 p. 255 that

$$\lim_{n\to+\infty} \int_\Gamma S_n(z)\ dz = \int_\Gamma S(z)\ dz$$

Hence

$$\sum_{j=0}^{+\infty} \left(\int_\Gamma g_j(z)\ dz \right) = \int_\Gamma \left(\sum_{j=0}^{+\infty} g_j(z) \right) dz$$

A survey of Taylor series The best known Taylor series are collected in the list below. They can all be found somewhere in the book. The radius of convergence R is also given.

$$e^z = \sum_{j=0}^{+\infty} \frac{z^j}{j!},\ R = \infty.$$

$$\sin z = \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{z^{2k-1}}{(2k-1)!},\ R = \infty.$$

$$\cos z = \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k}}{(2k)!},\ R = \infty.$$

$$\text{Log}\ z = \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{(z-1)^j}{j},\ R = 1.$$

$$\text{Log}\ (1+z) = \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{z^j}{j},\ R = 1.$$

$$\frac{1}{1-z} = \sum_{j=0}^{+\infty} z^j,\ R = 1.$$

3 Problem session

Exercise A Cauchy's Generalized Integral Formulas and the Deformation Invariance Theorem.

1. Let Γ denote the square loop with vertices at the points $\pm(2 \pm 2i)$ oriented counterclockwise. Determine for all integers $n \geq 0$ the value of the integral

$$\int_\Gamma \frac{e^{-z}}{(z - i\frac{\pi}{2})^{n+1}}\ dz$$

2. Determine the value of the integrals

$$\int_{|z|=2} \frac{\cos z}{z(z^2+9)}\ dz \quad \text{and} \quad \int_{|z-3i|=4} \frac{\cos z}{z(z^2+9)}\ dz$$

(Hint: Compare with **Exercise B (4)** in the 8th problem sheet.)

Exercise B Application of Liouville's Theorem.
Let f be an entire function, i.e. a function which is analytic in the entire complex plane $\mathbb{C}$. Assume there is a positive constant M such that Re $f(z) \leq M$ for all z in $\mathbb{C}$, hence that the real part of f is bounded from above. Show that f is constant. (Hint: Consider the function defined as $F(z) = e^{f(z)}$, and show first that F is constant; then show that f is constant.)

Exercise C The Maximum Modulus Principle.

Given the functions

$$f(z) = e^{z^2} \quad \text{and} \quad g(z) = z^2 - \tfrac{1}{4}$$

1. Find the maximum modulus of f and the maximum modulus of g in the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$.
2. Set

$$F(z) = \frac{1}{f(z)} \quad \text{and} \quad G(z) = \frac{1}{g(z)}$$

   For which of these functions is it possible to apply the Maximum Modulus Principle in $\overline{\mathbb{D}}$? In that case interpret this as a minimum modulus result for the original function.

Exercise D Application of Cauchy's Estimates.

Let f be an entire function. Assume there is a positive constant K such that $|f(z)| \leq K|z|$ for all $z \in \mathbb{C}$.

1. Show that $f''(z) = 0$ for all $z \in \mathbb{C}$. (Hint: Choose an arbitrary point z_0 in $\mathbb{C}$, and consider an arbitrary circle $C_R : \ |z - z_0| = R$. First determine an estimate $|f(z)| \leq M_R$, which is satisfied for all $z \in C_R$ and then an estimate of $|f''(z_0)|$. Finally let R tend to $+\infty$.)
2. Show that there exists a complex number a so that $f(z) = az$.
3. Let g be an entire function. Assume there is a positive constant L such that $|g(z)| \leq L|z|^2$ for all sufficiently large values of $|z|$, i.e. for all $|z| \geq r_0$ for a suitable $r_0 > 0$. Show that g is a polynomial of degree at most 2. (Hint: Show that $g^{(3)}(z) = 0$ for all $z \in \mathbb{C}$.)
4. How can these results be generalized ?

Exercise E Application of the Fundamental Theorem of Algebra.

Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0$$

denote a polynomial of degree n, and let $z_1, z_2, \ldots, z_r$ denote the different roots of multiplicity $d_1, d_2, \ldots, d_r$ respectively. Hence

$$P(z) = a_n (z - z_1)^{d_1} (z - z_2)^{d_2} \cdots (z - z_r)^{d_r}$$

1. Show that

$$\frac{P'(z)}{P(z)} = \frac{d_1}{z - z_1} + \frac{d_2}{z - z_2} + \cdots + \frac{d_r}{z - z_r}$$

2. Let Γ be a simple closed positively oriented curve, not passing through any of the roots of P. Show that

$$\frac{1}{2\pi i} \int_\Gamma \frac{P'(z)}{P(z)}\, dz \; = \; \text{the number of roots inside } \Gamma \text{ counted with multiplicity}$$

Exercise F Estimates of absolute values of rational functions for $|z|$ sufficiently large.

Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0$$

denote a polynomial of degree n. For $z \neq 0$

$$P(z) = z^n \left(a_n + a_{n-1}/z + \cdots + a_1/z^{n-1} + a_0/z^n \right)$$

1. Set $K = |a_n| + |a_{n-1}| + \cdots + |a_0|$. Show that

$$|P(z)| \ \leq \ K|z|^n \quad \text{for} \quad |z| \geq 1$$

2. Show that for R sufficiently large, there exists a positive constant L such that

$$|P(z)| \ \geq \ L|z|^n \quad \text{for} \quad |z| \geq R$$

(Hint: Use for instance $L = |a_n|/2$ and compare with (2) and (i) p. 216.)
Comment: It is possible to give a precise estimate of what "sufficiently large" could be, but it is irrelevant here. More important is it to understand what is meant by stating that an inequality is satisfied for all z sufficiently large (in absolute value).

3. Let $f(z) = P(z)/Q(z)$ where Q is a polynomial of degree m. Show that for R sufficiently large, there exists a positive constant M such that

$$\left|\frac{P(z)}{Q(z)}\right| \ \leq \ \frac{M}{|z|^p} \quad \text{for} \quad |z| \geq R \text{ , where } p = m - n.$$

Estimates of this type play a crucial role in section 6.3, in particular when $m > n$.

4 Homework problems

On Thursday, November 13 solutions to the problems will be posted on the course homepage.

1. § 4.4 **Exercise 18 and 19. Estimate of integrals, once more.**
Hint: In Exercise 19, set $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $a_n \neq 0$ and $n \geq 2$, and use an estimate of $|P(z)|$ similar to the one stated in Exercise F (2) from the problem session on this sheet.

2. § 4.5 **Exercise 7. Cauchy's Generalized Integral Formulas.**

3. § 4.6 **Exercise 8. The Maximum Modulus Principle.**