Complex Analysis 01141

Department of Mathematics

Week 10, 2008

1 Coverage next week

In the **11th week** we finish chapter 5 on infinite series with §§ $5.5 - 5.7$. We focus on Laurent series of functions that are analytic in annuli, and also on zeros of analytic functions and different types of isolated singularities at z_0 in $\mathbb{C}$ or at ∞.

2 Comments on the material for next week

Main theorem Let f be analytic in an annulus $A_{r,R}$: $r < |z - z_0| < R$. The Laurent Theorem (Theorem 14 pp. 269 - 270) expresses that f is equal to a Laurent series of the form

$$\sum_{j=-\infty}^{+\infty} a_j(z - z_0)^j = \sum_{j=0}^{+\infty} a_j(z - z_0)^j + \sum_{j=1}^{+\infty} a_{-j}(z - z_0)^{-j} \quad \text{for all } z \in A_{r,R}$$

The series on the left converges (by definition) in $A_{r,R}$ when both of the series on the right converge in $A_{r,R}$. Since a power series converges in a disc, the series $\sum_{j=0}^{+\infty} a_j(z-z_0)^j$ converges for all $|z-z_0| < R$, and the series $\sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$ converges for all $r < |z-z_0|$. The last statement follows from the fact that the related power series $\sum_{j=1}^{+\infty} a_{-j}\zeta^j$ must converge for all $\frac{1}{R} < |\zeta| < \frac{1}{r}$ and hence in the disc $|\zeta| < \frac{1}{r}$.
An analytic function has different Laurent expansions in different annuli, but the expansion in each annulus is uniquely determined since the coefficients satisfy equation (1) p. 269. Both series are uniformly convergent in any closed annulus of the form $\rho_1 \leq |z - z_0| \leq \rho_2$ where $r < \rho_1 < \rho_2 < R$.

Isolated singularities in C and at ∞ A function f has an isolated singularity at z_0 in $\mathbb{C}$ if f is analytic in some punctured disc around z_0. It has an isolated singularity at ∞ if f is analytic for all z of absolute value greater than some R. There are three types of isolated singularities: removable, pole of order m, and essential. The type of an isolated singularity z_0 in $\mathbb{C}$ is defined by the type of the Laurent expansion of f in a punctured disc of z_0 (see Definition 8 pp. 278 - 279); the type is also determined by the different properties expressed in Theorem 18 p. 284. The type of an isolated singularity of f at ∞ is by definition the same as the type of the isolated singularity of $F(\zeta) = f(1/\zeta)$ at $\zeta = 0$.

Special isolated singularities Example 2 p. 281 is formulated for rational functions $\frac{P(z)}{Q(z)}$. It is valid in greater generality for functions of the form $f(z) = \frac{g(z)}{h(z)}$ where both g and h are analytic at the point z_0 and where also g is allowed to have a zero at z_0 . Assume h has a zero of order $m \geq 1$ at z_0 . Then z_0 is an isolated singularity of f since f is analytic in a punctured disc around z_0. Note that

$$h(z) = (z - z_0)^m h_1(z) \text{ where } h_1 \text{ is analytic at } z_0 \text{ and } h_1(z_0) \neq 0$$

Similarly we express g as

$$g(z) = (z - z_0)^n g_1(z) \text{ where } g_1 \text{ is analytic at } z_0 \text{ and } g_1(z_0) \neq 0$$

i.e. if g has a zero at z_0 then it is of order n, and if $g(z_0) \neq 0$ then $n = 0$ and $g(z) = g_1(z)$. It follows that

$$f(z) = \frac{g(z)}{h(z)} = (z - z_0)^{n-m} \frac{g_1(z)}{h_1(z)}$$

Hence, if $n < m$ then f has a pole of order $m - n$ at z_0. If on the other hand, $n \geq m$ then f has a removable singularity at z_0 and f can be extended to a function F which is analytic at z_0 by removing the singularity, i.e. by defining $F(z_0) = \lim_{z \to z_0} f(z) = \frac{g_1(z_0)}{h_1(z_0)}$. If $n > m$ then the point z_0 is a zero of order $n - m$ of the function F.

Radius of convergence, isolated singularities and branch points On p. 256 it says that the radius of convergence R of the Taylor series of an analytic function $f : D \to \mathbb{C}$ around $z_0 \in D$ always is equal to the radius of the largest disc in D centred at z_0. This is only correct if the domain of analyticity of f is interpreted correctly.

Consider for instance the principal branch of the logarithm Log and its Taylor series around $z_0 = -4 + 3i$. The largest disc centred at z_0 within the domain of analyticity of Log has radius 3. But the radius of convergence of the series is $5 = |z_0|$. That the radius of convergence is greater than or equal to 5 follows from the fact that the Taylor series of Log around the given z_0 is identical to the Taylor series of $\mathcal{L}_0$ around z_0 and $\mathcal{L}_0$ is analytic in the slit plane $\mathbb{C} \setminus [0, +\infty[$. On the other hand the radius of convergence can not be greater than 5 since $\ln |z| \to -\infty$ as $z \to 0$.

Consider also $f(z) = (z^2 - \frac{\pi^2}{4}) \tan z$ with isolated singularities at $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. The radius of convergence of the Maclaurin series of f is equal to $\frac{3\pi}{2}$ and not $\frac{\pi}{2}$ since f has removable singularities at $\pm\frac{\pi}{2}$ and the Maclaurin series of f is identical to the Maclaurin series of the analytic function obtained when these singularities are removed. The radius of convergence cannot be greater than $\frac{3\pi}{2}$ since both functions have poles at $\pm\frac{3\pi}{2}$ and the functions tend to ∞ as z tends to $\pm\frac{3\pi}{2}$.

One can be sure to have found the maximal domain of convergence of a Taylor series (or of a Laurent series) if one can find a point at the bounding circle (or points at the bounding circles) which prevents further convergence. Such points are typically poles or essential singularities, or branch points like 0 for branches of the logarithm and functions derived from those. But removable singularities do not prevent convergence.

3 Problem session

Exercise A Geometric series.

1. State for which z in $\mathbb{C}$ the series

$$\sum_{j=0}^{+\infty} (z - 1)^j$$

is convergent and for which z the series is divergent. Determine the sum of the series when convergent.

2. Answer the same questions for the series

$$\sum_{j=0}^{+\infty} \frac{1}{(z-1)^j} \quad \text{and} \quad \sum_{j=1}^{+\infty} \frac{1}{(z-1)^j}$$

Notice the difference in the ranges for the summation index j.

3. Answer the same questions for the series

$$\sum_{j=0}^{+\infty} e^{jz}$$

4. Answer the same questions for the series

$$\sum_{j=0}^{+\infty} \left(\frac{z-1}{z+1} \right)^j$$

Exercise B Domains of convergence.

Determine for each of the following series the maximal domain in which the series is absolutely convergent:

$$\sum_{j=1}^{+\infty} \frac{1}{j!\; z^j} \;; \qquad \sum_{j=1}^{+\infty} \frac{j}{j+1}\,(2z)^j \;; \qquad \sum_{j=1}^{+\infty} j!\, \left(\frac{z}{j}\right)^j$$

(Recall that $(1+\frac{1}{j})^j \to e$ as $j \to +\infty$.)
State which of the series are power series and determine the radius of convergence of those. Radius of convergence is only defined for power series.

Exercise C Taylor series.

Consider the function

$$f(z) = \frac{1}{3-z}$$

1. Determine the Maclaurin series of f, and state its radius of convergence.
   (Hint: Rewrite $\frac{1}{3-z} = \frac{1}{3}\frac{1}{1-z/3}$ and use $\sum_{j=0}^{+\infty} z^j = \frac{1}{1-z}$ for $|z| < 1$.)
   State for all $j = 0, 1, 2, \ldots$ the coefficients a_j of the Maclaurin series. (Recall that a_j is the coefficient of z^j, or in a Taylor series around z_0 the coefficient of $(z-z_0)^j$.)
2. Determine the Taylor series of f around $z_0 = 1$, and state its radius of convergence.
   (Hint: Start by rewriting $\frac{1}{3-z} = \frac{1}{(3-1)-(z-1)}$ and continue as above.)
   State for all $j = 0, 1, 2, \ldots$ the coefficients a_j of the Taylor series.

Exercise D Cauchy Multiplication.

Consider the functions

$$g(z) = \frac{z}{1+e^z} \quad \text{and} \quad h(z) = \frac{\sin z}{1+z^2}$$

1. Determine the four coefficients a_j, $j = 0, 1, 2, 3$, in the Maclaurin series of g. (Hint: Compare with Example 5 p. 249.)
2. Determine the Maclaurin series of h. (Hint: Compare with Example 4 pp. 248 - 249 or Example 5 p. 249.)

Exercise E Uniform convergence.

Consider the geometric series

$$\sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j}$$

1. Determine the domain of convergence of the series.
2. Explain why the series converges uniformly along the circle $|z| = 10$. (Hint: Apply for instance Theorem 7 p. 253.)
3. Calculate

$$\int_{|z|=10} \left(\sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j} \right) dz$$

   (Hint: Set $f_n(z) = \sum_{j=0}^{n} \frac{3^j}{(z-i)^j}$ and $f(z) = \sum_{j=0}^{+\infty} \frac{3^j}{(z-i)^j}$ and apply Theorem 8 p. 255.)

4 Homework problems

Solutions to the homework problems will be posted on the course homepage no later than November 19.

The exam problems can be found on the homepage.

1. **Exam January 1997, Exercise 3.**
2. **Exam January 2000, Exercise 5.**
3. **Exam December 2003, Exercise 3.**