Complex Analysis 01141

Department of Mathematics

Week 11, 2008

1 Coverage next week

In the **12th week** we focus on §§ 6.1 and 6.3 : residues and Cauchy's Residue Theorem as well as improper integrals.

2 Comments on the material for next week

Calculating residues If the Laurent series of f is known in a punctured disc of an isolated singularity z_0 then the residue $\text{Res}(f; z_0)$ is determined immediately (by definition) as the coefficient a_{-1} to $(z - z_0)^{-1}$.
At an essential singularity the residue can only be determined in this way.
At a removable singularity the residue is equal to 0.
At a simple pole the residue can be determined by the method in formula (3) p. 309 or the method stated in Example 2 p. 309. Note, that in Example 2 the functions P and Q are not assumed to be polynomials, the example should rather have been formulated in general notation $f(z) = \frac{g(z)}{h(z)}$. For a pole of order m the residue can be determined by the formula (4) given in Theorem 1 p. 310.

Cauchy's residue theorem The Cauchy Residue Theorem (Theorem 2 p. 312) is one of the main theorems in complex analysis. Note that it contains Cauchy's Integral Formula (Theorem 14 p. 204) and Cauchy's Generalized Integral Formulas (Theorem 19 p. 211) as special cases. Under the given assumptions the value of the integral of f along the loop Γ is determined in formula (5) p. 312 by the value of the residues of the finitely many isolated singularities inside Γ.

Improper integrals In § 6.3 are given examples of how Cauchy's Residue Theorem can be used to determine the value of some improper integrals of real or complex functions.
As explained on p. 318 f is said to be *integrable* over $]-\infty, +\infty[$ if the limits of both integrals $\int_c^0 f(x)\, dx$ and $\int_0^b f(x)\, dx$ exist as $c \to -\infty$ and $b \to +\infty$ respectively, and the improper integral $\int_{-\infty}^{+\infty} f(x)\, dx$ is said to be *convergent* in this case. If f is integrable over $]-\infty, +\infty[$ then

$$\int_{-\infty}^{+\infty} f(x)dx = \text{ p.v. } \int_{-\infty}^{+\infty} f(x)dx$$

However, the principal value (p.v.) can exist without f being integrable, as explained in the text on p. 319 above Example 1.
The methods used for determining the value of an improper integral are not formulated in a theorem, but are illustrated in the Examples 1, 2, and 3 in § 6.3. In all three examples we consider a family of positively oriented simple closed curves Γ_ρ depending on a real parameter ρ. Note that in the examples the value $V(\rho)$ of the integral along Γ_ρ does not depend on ρ for ρ sufficiently large, hence $V = V(\rho)$ is constant.
In Example 1 and 2 it is shown that the integral along C_ρ^+ tends to 0 as ρ tends to $+\infty$. The principal value of the integral over the real axis is therefore equal to V.

In Example 3 it is shown that the integrals along both γ_2 and γ_4 tend to 0 as ρ tends to $+\infty$. Hence, the sum of the integrals along γ_1 and γ_3 tends to the constant value V as ρ tends to $+\infty$.
Note that if $f(x)$ and V take complex values, then we have found the principal value of two improper integrals

$$\text{p.v.} \int_{-\infty}^{+\infty} \operatorname{Re} f(x)dx \; = \operatorname{Re} V \quad \text{and} \quad \text{p.v.} \int_{-\infty}^{+\infty} \operatorname{Im} f(x)dx \; = \operatorname{Im} V$$

Moreover, if f is an even function, i.e. $f(-x) = f(x)$, for which the principal value of the integral over the real axis exists, we have $\int_{-\rho}^{0} f(x)\ dx = \int_{0}^{+\rho} f(x)\ dx$ and therefore

$$\int_0^{+\infty} f(x)dx = \tfrac{1}{2}\left(\text{p.v.} \int_{-\infty}^{+\infty} f(x)dx\right)$$

3 Problem session

Exercise A Laurent series, obtained from geometric series.
Consider as in **Exercise C**, from the 10th problem sheet, the function

$$f(z) = \frac{1}{3-z}$$

1. Determine the Laurent series of f in the annulus $|z| > 3$.
(Hint: Rewrite $\frac{1}{3-z} = -\frac{1}{z}\frac{1}{1-3/z}$ and use $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$ for $|z| < 1$. Compare also with example 1 and 2 pp. 273 - 275.)
State the coefficients a_j for arbitrary $j \in \mathbb{Z}$. (Recall that a_j with index $j \in \mathbb{Z}$ is the coefficient to z^j. In the case of a Laurent series around z_0, a_j is the coefficient to $(z - z_0)^j$.)
2. Determine the Laurent series of f in the annulus $|z - 1| > 2$. State the coefficients a_j for arbitrary $j \in \mathbb{Z}$.
3. Determine the Laurent series of f in the annulus $|z - 3| > 0$. State the coefficients a_j for arbitrary $j \in \mathbb{Z}$.

Exercise B Laurent series, obtained from known Maclaurin series.
Given

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right) \quad \text{and} \quad g(z) = \frac{e^z}{(z+1)^2}$$

1. Determine the Laurent series of f in the annulus $|z| > 0$. State the coefficients for arbitrary j.
2. Determine the Laurent series of g in the annulus $|z + 1| > 0$. (Hint: Introduce the auxiliary function e^{z+1}.) State the coefficients for arbitrary j.
3. Let C denote the circle $|z| = 3$. State the value of the integrals

$$\int_C f(z)\ dz \qquad \text{and} \qquad \int_C g(z)\ dz$$

Exercise C Isolated singularities.
Determine in each of the cases below whether z_0 is an isolated singularity of f, and if so state its type: removable, pole, or essential. For a pole the order must be given. In case of a removable singularity determine also $\lim_{z\to z_0} f(z)$.

$$(\mathbf{1}) \quad f(z) = \frac{iz+1}{z-2} \quad \text{and} \quad z_0 = 2\ ; \qquad (\mathbf{2}) \quad f(z) = \frac{iz+1}{z-2} \quad \text{and} \quad z_0 = \infty\ ;$$

$$(\mathbf{3})\quad f(z) = \frac{z}{(z^2+1)^3} \quad \text{and} \quad z_0 = i\ ; \qquad (\mathbf{4})\quad f(z) = \frac{\cos z - 1}{z^2} \quad \text{and} \quad z_0 = 0\ ;$$

$$(\mathbf{5})\quad f(z) = \sin\left(\frac{1}{z}\right) \quad \text{and} \quad z_0 = 0\ ; \qquad (\mathbf{6})\quad f(z) = \sin z \quad \text{and} \quad z_0 = \infty\ ;$$

$$(\mathbf{7})\quad f(z) = \frac{e^z - 1}{z^2} \quad \text{and} \quad z_0 = 0\ ; \qquad (\mathbf{8})\quad f(z) = \frac{1}{\sin(1/z)} \quad \text{and} \quad z_0 = 0\ .$$

Exercise D Cauchy Multiplication of Laurent series.
Given $f(z) = \cot z$.

1. Determine all zeros and all isolated singularities of f, and state their type.
2. The function f has infinitely many Laurent expansions around $z = 0$ in infinitely many different annuli of the form $A_{R_1,R_2} = \{\ z \in \mathbb{C} \mid R_1 < |z| < R_2\ \}$. However, in each of the annuli A_{R_1,R_2} where f is analytic the expansion is unique. State the largest possible annulus of the form $0 < |z| < R$. Moreover, determine the part $a_{-1}z^{-1} + a_0 + a_1 z$ of the Laurent expansion of f in this annulus.

Exercise E Radius of convergence.
Compare with the remarks on the 10th weekly worksheet concerning radius of convergence, isolated singularities, and branch points.

1. State the radius of convergence of the Maclaurin series of the function
$$f(z) = \frac{4z^2 - \pi^2}{\cos z}$$
without determining the series.
2. State the radius of convergence of the Taylor series around $z_0 = -1 - i$ for the principle branch of the logarithm Log.

Exercise F An incorrect argument.
What is wrong with the following argument:
Since
$$\sum_{j=0}^{+\infty} z^j = \frac{1}{1-z}$$
and
$$\sum_{j=1}^{+\infty} \frac{1}{z^j} = \frac{1}{z-1}$$
we have
$$\sum_{-\infty}^{+\infty} \frac{1}{z^j} = \frac{1}{z-1} + \frac{1}{1-z} = 0$$

4 Homework problems

On Thursday, November 27 solutions to the problems will be posted on the course homepage.

1. § 5.2 **Exercise 11 (a) and (b). Cauchy Multiplication.**
Compare with the Examples 4 and 5 pp. 248 - 249.

2. § 5.3 **Exercise 6. A Maclaurin series.**
Comment: In § 5.6 we define *isolated singularities* of analytic functions. The function $\frac{\sin z}{z}$ has an isolated singularity at 0. The singularity is *removable.* The function f is precisely the analytic function one obtains after having removed the singularity at 0.

3. § 5.5 **Exercise 9. All Laurent series around $z_0 = 0$.**
   Extend the exercise by adding:
   Let the annulus of convergence for the given Laurent series be of the form

$$A_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\} \ .$$

   For each $z \in A_{r,R}$ the sum of the series has a value that we denote by $f(z)$. Determine the sum function f of the Laurent series in $A_{r,R}$. The function is in fact defined and analytic in all of $\mathbb{C}$ except two points.
   Determine the Maclaurin series of f in the disc $|z| < r$. State the coefficients.
   Determine the Laurent series of f in the annulus $|z| > R$. State the coefficients.

4. § 5.6 **Exercise 1. Isolated singularities and their type.**
   On p. A-40 the answer to **(h)** concerning $z = 0$ is wrong because 0 is not an isolated singularity.