Complex Analysis 01141

Department of Mathematics

Week 13, 2008

1 Exam

The exam takes place on December 12. The text will be in English. Your answers may be given in Danish or in English. Remember that reasons must be given for all answers.

Solutions to the set of exam problems will be posted on the course homepage after the exam has ended.

The exam requirements can be found on the course homepage.

Material on the homepage Problem sheets, weekly sheets, sets of former exam problems, solutions to homework problems and short answers to former exam problems.

Office hours prior to the exam My office is in room 046 in the East wing of Building 303 South.

2 Problem session

Exercise A Given the function $f(z) = x^2 + iy^2$, where x and y are the real and imaginary parts of z. Determine the points z where f is differentiable.
State the derivative $f'(z)$ at each of these points. Explain why f is not analytic at any point.

Exercise B Let f denote the Möbius transformation

$$f(z) = \frac{1+z}{1-z}$$

and let

$$\mathbb{D}_+ = \{z \in \mathbb{C} \mid |z| < 1 \ \wedge \ \text{Im } z > 0\}$$

denote the upper semi-disc.

1. Determine the image $f(\mathbb{D}_+)$.
2. Solve the Dirichlet problem

$$\phi : \overline{\mathbb{D}_+} \to \mathbb{R}$$

satisfying the boundary condition

$$\phi(z) = \begin{cases} -1 & \text{for } |z| = 1 \text{ and Im } z > 0 \\ 0 & \text{for } z \in]-1, +1[\end{cases}$$

Exercise C Consider the function

$$f(z) = \frac{z}{e^z - 1}$$

Explain why f can be expressed as a power series

$$f(z) = \sum_{j=0}^{+\infty} a_j z^j$$

in a punctured disc $0 < |z| < R$ for suitable R.Determine the radius of convergence R of the power series. The so-called *Bernoulli numbers* B_j are related to the coefficients in this series and defined as $a_j = \frac{B_j}{j!}$. Determine B_0, B_1, and B_2 .

Exercise D Consider the function

$$f(z) = \frac{1}{z^3(1+z^2)}$$

1. Determine the Laurent series of f in the maximal annulus of the form $0 < |z| < R$, and state R .
2. Determine the Laurent series of f in the annulus $|z| > R$.

Exercise E Consider the function

$$f(z) = \frac{e^{iz}}{\cosh z} \quad \text{where} \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

and for $\rho > 0$ the rectangular loop

$$\Gamma_\rho = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

with vertices at $\pm\rho$ and $\pm\rho + \pi i$. The line segments γ_j , $j = 1, 2, 3, 4$, are numbered so that γ_1 corresponds to the line segment from $-\rho$ to ρ , γ_2 to the one from ρ to $\rho + \pi i$, γ_3 to the one from $\rho + \pi i$ to $-\rho + \pi i$ and γ_4 to the one from $-\rho + \pi i$ to $-\rho$. Sketch the loop.

1. Show that
$$\int_{\Gamma_\rho} f(z)\ dz = 2\pi e^{-\frac{\pi}{2}}$$
2. Show that the integrals along γ_2 and γ_4 tend to 0 as ρ tends to $+\infty$. (Hint: You may use the formula $|\cosh(x + iy)| = \sqrt{\sinh^2 x + \cos^2 y}$.)
3. Determine the value of the improper integral
$$\int_0^{+\infty} \frac{\cos x}{\cosh x}\ dx$$
Incidentally, this is an integral Maple 12 cannot do.

Exercise F Consider the functions

$$f(z) = \frac{1+z}{1-z}\ , \quad g(z) = z^2, \quad h(z) = \text{Log}\ z$$

where Log denotes the principal branch of the logarithm.

1. State where g is conformal. The function g is mapping the plane $\mathbb{C}$ onto itself, but the function is not bijective. Give examples of sub-domains E of $\mathbb{C}$ so that g is mapping E bijectively and conformally onto $g(E)$.
2. Set
$$A = \{\ z \in \mathbb{C} \mid -\pi < \text{Im}\ z < \pi\ \} \quad \text{and} \quad \mathbb{D} = \{\ z \in \mathbb{C} \mid |z| < 1\ \}.$$
Determine $C = f(\mathbb{D})$, $B = g(C)$, and show that $A = h(B)$. Explain why $F = h \circ g \circ f$ is a bijective conformal mapping of $\mathbb{D}$ onto A. State the inverse functions $h^{-1} : A \to B$, $g^{-1} : B \to C$ and $f^{-1} : C \to \mathbb{D}$.
3. Describe the family of curves in $\mathbb{D}$ which by F is mapped onto the family of straight lines in A parallel to the real axis. Sketch these curves in $\mathbb{D}$.